

Partially Ordered Sheaves on a Locale. I ^{*}

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Abstract

In this paper, we investigate the order algebraic structure in the category of sheaves on a given locale X . Since every localic topos has a generating set formed by its subterminal objects, we define a “point” of a partially ordered sheaf to be a morphism from a subterminal sheaf to the partially ordered sheaf. Using the concept of “points”, we investigate the completeness of posheaves systemically. Some internal characterizations of complete partially ordered sheaves and frame sheaves are given. We also give an explicit description of the construction of associated sheaf locales and show directly that the category $Sh(X)$ of sheaves on a locale X is equivalent to the slice category LH/X of locales and local homeomorphisms over X . Applying this equivalence, we give characterizations of partially ordered sheaves and complete partially ordered sheaves in terms of sheaf locale respectively.

Keywords: sheaf; partially ordered sheaf; sheaf locale.

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1 Introduction

The theory of partially ordered sets in a topos has been studied extensively (see [1], [2], [3]). When we restrict our attention to a localic topos, i.e. the topos $Sh(X)$ of sheaves on a given locale X , it is interesting to investigate the properties of ordered sheaves. In this paper, we first introduce the concept of partially ordered sheaves on a given locale X which are partially ordered objects in the localic topos $Sh(X)$. For every localic topos $Sh(X)$, it is well known that it is not well-pointed, i.e. the terminal sheaf 1 can not generate $Sh(X)$. But the localic topos $Sh(X)$ has a generating set formed by its subterminal objects. This implies that in a localic topos, there exists enough “points” of morphisms from subterminal sheaves to a sheaf F such as $\hat{1} \rightarrow F$, where $\hat{1}$ is a subsheaf of the terminal sheaf 1 . For a given sheaf F , these points in F will act similar

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as what points in a set. By using the concept of “points”, we define the concepts such as upper bound and *supremum* for a subsheaf of a partially ordered sheaf. This makes us can investigate those more complicated concepts such as complete partially ordered sheaves and frame sheaves.

The paper is organized in to five sections. In section 2, we introduce the concept of partially ordered sheaves, and investigate the basic properties of partially ordered sheaves. In section 3, we investigate the completeness of partially ordered sheaves, an internal characterization of complete partially ordered sheaves is given. In particular, we give a characterization of complete Heyting sheaves on X which are properly internal frames in the localic topos $Sh(X)$. At the end, we give an explicit description of the associated sheaf locale of a given sheaf and show directly that the category $Sh(X)$ of sheaves on a locale X is equivalent to the slice category LH/X of locales and local homeomorphisms over X . Thus characterizations of partially ordered sheaf locales, and complete partially ordered sheaf locales are presented respectively. Throughout this paper, when we write X for a locale, we will write $\mathcal{O}(X)$ for the corresponding frame. Readers may refer to [5] for notations and terminology not explicitly given here.

2 Partially Ordered Sheaves

Definition 2.1. Let X be a locale and F a sheaf on X . F is called a partially ordered sheaf (shortly posheaf) if and only if F satisfies the following conditions:

- (POS1) $F(u)$ is a partially ordered set for every $u \in \mathcal{O}(X)$;
- (POS2) every restriction map $F(u) \rightarrow F(v)$ for $v \leq u$ is order-preserving;
- (POS3) given two compatible families $\{s_i \in F(u_i) \mid i \in I\}$ and $\{t_i \in F(u_i) \mid i \in I\}$ with $s_i \leq t_i$ for any $i \in I$. If $\{s_i \in F(u_i) \mid i \in I\}$ patch to an element $s \in F(\bigvee u_i)$ and $\{t_i \in F(u_i) \mid i \in I\}$ patch to another element $t \in F(\bigvee u_i)$, then $s \leq t$.

Examples of partially ordered sheaves are numerous, for example the continuous real-valued function sheaf C which sends each open $u \in \mathcal{O}(X)$ the set $C(u) = \{f : u \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$ of continuous real-valued functions on u with pointwise order is a posheaf on X . But a sheaf of partially ordered sets in general does not satisfy the condition (POS3), hence not always a posheaf.

Example 2.1. Consider the continuous real-valued function sheaf C on X . If we keep the pointwise order on $C(u)$ for $u \neq 1_X$ and take discrete order on $C(1_X)$ then it is a sheaf of partially ordered sets but not a posheaf.

Lemma 2.1. F is a posheaf iff F is a sheaf of partially ordered sets and satisfies (POS3).

If F is a posheaf on X , then the sheaf F^{op} defined by $F^{op}(u) = F(u)^{op}$, the opposite poset of $F(u)$, and each restriction map $F^{op}(u) \rightarrow F^{op}(v)$ is same as $F(u) \rightarrow F(v)$, is

a poshaf. If G is a subsheaf of F , it is clear that G is a posheaf for the induced order. Recall that an internal partially ordered object in a topos ε is an object A of ε with a subobject $\leq_A \rightharpoonup A \times A$ satisfying the following conditions:

- (i) (reflexivity) the diagonal $\delta : A \rightarrow A \times A$ can be factored through $\leq_A \rightharpoonup A \times A$;
- (ii) (antisymmetry) the intersection $\leq_A \cap \leq_A^{op}$ of subobjects is contained in the diagonal, where \leq_A^{op} is the image of the composite $\leq_A \xrightarrow{l} A \times A \xrightarrow{t} A \times A$ with $A \times A \xrightarrow{t} A \times A$ the twist map interchange the factors of the product;
- (iii) (transitivity) the subobject $\langle p_1 v, p_2 u \rangle : C \rightharpoonup A \times A$ can be factored through $\leq_A \xrightarrow{l} A \times A$, where C is defined as the following pullback with projections u and v :

$$\begin{array}{ccc} C & \xrightarrow{u} & \leq_A \\ v \downarrow & & \downarrow p_1 \\ \leq_A & \xrightarrow{p_2} & A \end{array}$$

If F is a posheaf, then the sub-presheaf $\leq_F : \mathcal{O}(X)^{op} \rightarrow \text{Set}$ of the product sheaf $F \times F$ defined by $\leq_F(u) = \{(x, y) \in F(u) \times F(u) \mid x \leq y\}$ for $u \in \mathcal{O}(X)$ is a sheaf and satisfying conditions (i)-(iii). Conversely, if we have a subsheaf $\leq_F \rightharpoonup F \times F$ satisfying conditions (i)-(iii), then F is a posheaf. So we have the following result.

Lemma 2.2. *F is a posheaf on a locale X if and only if F is an internal partially ordered object in the localic topos $Sh(X)$.*

We know that a localic topos is in general not well-pointed, i.e. the terminal object 1 is not a generator. But every localic topos can be generated by the subobjects of its terminal object 1 . This implies that in a topos $Sh(X)$ of sheaves, those “points” $\hat{1} \rightarrow F$ of a sheaf F can act somewhat as points in the category of sets. We define a point of a sheaf F to be a morphism $p : \hat{1} \rightarrow F$ with $\hat{1}$ a subsheaf of the terminal sheaf 1 . A point of the form $1 \rightarrow F$ will be called a global point of F . For a point $p : \hat{1} \rightarrow F$ of F , we write $\text{dom}(p)$ for the largest open $u \in \mathcal{O}(X)$ with $p(u) \neq \emptyset$, i.e. $\text{dom}(u) = \bigvee \{u \in \mathcal{O}(X) \mid p(u) \neq \emptyset\}$, and call it the domain of p . If we look at the image of a point $p : \hat{1} \rightarrow F$, a point p of F can be equivalently regarded as an element of $F(\text{dom}(p))$. The set of all points of a sheaf F will be denoted by F_p .

Definition 2.2. Let F be a posheaf. We define a partial order on the set F_p as following

$$p_1 \leq p_2 \Leftrightarrow \text{dom}(p_1) \leq \text{dom}(p_2), \text{ and } p_1(\text{dom}(p_1)) \leq p_2(\text{dom}(p_2))|_{\text{dom}(p_1)}$$

where $p_2(\text{dom}(p_2))|_{\text{dom}(p_1)}$ be the restriction of $p_2(\text{dom}(p_2))$ on $\text{dom}(p_1)$.

This definition of partial order is equivalent to saying that for two points $p_1 : \hat{1} \rightarrow F$ and $p_2 : \check{1} \rightarrow F$, $p_1 \leq p_2$ if and only if there is a morphism $h : \hat{1} \rightarrow \check{1}$ such that $\langle p_1, p_2 h \rangle : \hat{1} \rightarrow F \times F$ can be factored through $\leq \rightharpoonup F \times F$.

Given a morphism $\alpha : F \rightarrow G$ of sheaves, we have a natural map $F_p \rightarrow G_p, p \mapsto \alpha p$. For morphisms $\alpha : F \rightarrow G$ and $\beta : F \rightarrow G$ of posheaves, we define $\alpha \leq \beta$ if and only if $\alpha p \leq \beta p$ for all points $p \in F_p$. Then we have

Lemma 2.3. *For morphisms $\alpha : F \rightarrow G$ and $\beta : F \rightarrow G$ of posheaves, the following are equivalent:*

- (1) $\alpha \leq \beta$;
- (2) $\forall u \in \mathcal{O}(X), \forall x \in F(u), \alpha_u(x) \leq \beta_u(x)$;
- (3) the diagonal $\langle \alpha, \beta \rangle : F \rightarrow G \times G$ can be factored through $\leq_G \rightarrow G \times G$.

Definition 2.3. Let F and G be partially ordered sheaves on a locale X , and $\alpha : F \rightarrow G$. α is called order-preserving if for any two points p_1, p_2 of F , $p_1 \leq p_2$ implies $\alpha p_1 \leq \alpha p_2$.

If $f : A \rightarrow B$ and $g : B \rightarrow C$ are order-preserving, it is clear that the composite $gf : A \rightarrow C$ is order-preserving.

Proposition 2.1. *Let F and G be partially ordered sheaves on a locale X , and $\alpha : F \rightarrow G$ a morphism. The following conditions are equivalent:*

- (1) α is order-preserving;
- (2) $\forall u \in \mathcal{O}(X), \alpha_u : F(u) \rightarrow G(u)$ is order-preserving;
- (3) the composite $\leq_F \rightarrow F \times F \xrightarrow{\alpha \times \alpha} G \times G$ can be factored through $\leq_G \rightarrow G \times G$, i.e. we have a morphism $\gamma : \leq_F \rightarrow \leq_G$ such that the following square commutes

$$\begin{array}{ccc} F \times F & \xrightarrow{\alpha \times \alpha} & G \times G \\ \uparrow & & \uparrow \\ \leq_F & \xrightarrow{\gamma} & \leq_G \end{array}$$

Definition 2.4. Two partially ordered sheaves F and G on a locale X is said to be order isomorphic if there is a isomorphism $\alpha : F \rightarrow G$ such that both α and its inverse α^{-1} is order-preserving.

Let F be a posheaf on X and G a subsheaf of F . We call G a downsheaf of F if for any two points $p, p' \in F_p$, $p \leq p'$ and $p' \in G_p$ implies $p \in G_p$.

Proposition 2.2. *Let F be a posheaf on X and G a subsheaf of F . The following conditions are equivalent:*

- (1) G is a downsheaf of F ;
- (2) $\forall u \in \mathcal{O}(X), G(u)$ is a downset of $F(u)$;
- (3) the classification map $\phi : F^{op} \rightarrow \Omega$ for G^{op} is order-preserving, where Ω be the subobject classifier in $Sh(X)$.

Proof The equivalence of (1) and (2) is clear.

(2) \Rightarrow (3) Condition (2) is equivalent to say that $G^{op}(u)$ is an upper set of $F^{op}(u)$ for every $u \in \mathcal{O}(X)$. For $u \in \mathcal{O}(X)$, $\phi_u : F^{op}(u) \rightarrow \Omega(u)$ is defined as $\phi_u(x) = \bigvee \{v \mid v \leq u, x|_v \in G(v)\}$, $x \in F(u)$.

$$\begin{array}{ccc} G(u) & \longrightarrow & 1(u) \\ \downarrow & & \downarrow \\ F^{op}(u) & \xrightarrow{\phi_u} & \Omega(u) \end{array}$$

So if $x \leq^{op} y$ in $F^{op}(u)$, i.e. $y \leq x$ in $F(u)$, then $x|_v \in G(v)$ implies $y|_v \in G(v)$. Hence $\phi_u(x) \leq \phi_u(y)$.

(3) \Rightarrow (2) If $x \leq y$ in $F(u)$ and $y \in G(u)$. Then $\phi_u(y) \leq \phi_u(x)$ and $\phi_u(y) = u$. Hence $\phi_u(x) = u$ this implies $x \in G(u)$. \square

Let F be a posheaf on X and p a point of F . We have a downsheaf $\downarrow p$ generated by p :

$$\downarrow p(u) = \begin{cases} \{x \in F(u) \mid x \leq p(u)\}, & p(u) \neq \emptyset \\ \emptyset, & p(u) = \emptyset \end{cases}$$

The downsheaf of the form $\downarrow p$ will be called a principle ideal of the posheaf F .

Dually, we define an uppersheaf of F to be a downsheaf of F^{op} . A special class of uppersheaves for which we will call principle filters has the form

$$\uparrow p(u) = \begin{cases} \{x \in F(u) \mid p(u) \leq x\}, & p(u) \neq \emptyset \\ \emptyset, & p(u) = \emptyset \end{cases}$$

where p is a point of F .

Let F be a sheaf on a locale X and $u \in \mathcal{O}(X)$. We have a sheaf F^u on the open u such that $F^u(v) = F(v)$ for every $v \leq u$. We call F^u the restriction of F to u . We some time also regard F^u as a subsheaf of F for which we take $F^u(w) = \emptyset$ for any $w \not\leq u$.

Recall the powersheaf $\mathbb{P}F$ of F defined by $\mathbb{P}F(u) = \text{Sub}(F^u)$ with restriction maps $\mathbb{P}F(u) \rightarrow \mathbb{P}F(v)$, $S \mapsto S^v$ for every $v \leq u$ in $\mathcal{O}(X)$. $\mathbb{P}F$ is a partially ordered sheaf for the subsheaves inclusion order. For every element $x \in F(u)$, we regard x as a point of F and thus have a downsheaf $\downarrow x$. Hence we have a morphism

$$\downarrow : F \rightarrow \mathbb{P}F$$

such that for every $u \in \mathcal{O}(X)$ and $x \in F(u)$, $\downarrow(u)(x) = \downarrow x$. We call $\downarrow : F \rightarrow \mathbb{P}F$ the principle ideal embedding. Similarly, we can construct a down-powersheaf $\mathbb{D}F$ of F such that $\mathbb{D}F(u) = \text{Dow}(F^u)$ where $\text{Dow}(F^u)$ is the set of all downsheaves of F^u with the same restriction maps with $\mathbb{P}F$. Then $\mathbb{D}F$ is a subsheaf of $\mathbb{P}F$, so we have an inclusion $\mathbb{D}F \hookrightarrow \mathbb{P}F$ and the principle ideal embedding $\downarrow : F \rightarrow \mathbb{P}F$ can be factored through $\mathbb{D}F$.

Now we consider the generalization of the important concept of Galois connection in classical order theory.

Definition 2.5. For order-preserving $\alpha : F \rightarrow G$ and $\beta : G \rightarrow F$, α is left adjoint to β (β is right adjoint to α), written $\alpha \dashv \beta$ if and only if the relations $\alpha x \leq y$ and $x \leq \beta y$ are equivalent for all points $x \in F_p$ and $y \in G_p$.

As in the standard case, an adjoint, if it exists, is uniquely defined and the adjointness can be characterized by $\alpha \dashv \beta$ if and only if $1_F \leq \beta\alpha$ and $\alpha\beta \leq 1_G$.

Similar to the case in the category of sets, an order-preserving morphism may have neither a left adjoint nor a right adjoint. But if an order-preserving morphism has right (left) adjoint then it is unique.

Lemma 2.4. $(\alpha \dashv \beta) : F \rightarrow G$ if and only if $\alpha_u : F(u) \rightarrow G(u)$ is left adjoint to $\beta_u : G(u) \rightarrow F(u)$ for every $u \in \mathcal{O}(X)$.

Now we construct a left adjoint for the inclusion $\mathbb{D}F \hookrightarrow \mathbb{P}F$.

If $S \in \text{Sub}(F)$ is a subsheaf of F , we write $\downarrow S$ the sub-presheaf of F such that $\downarrow S(u) = \{x \in F(u) \mid \exists \{u_i \mid i \in I\} \subset \mathcal{O}(X), x_i \in S(u_i), u = \bigvee u_i, x|_{u_i} \leq x_i, i \in I\}$ for $u \in \mathcal{O}(X)$. Then $\downarrow S$ is a downsheaf of F . Thus we have a morphism $\downarrow () : \mathbb{P}F \rightarrow \mathbb{D}F$ such that for every $u \in \mathcal{O}(X)$ and $S \in \mathbb{P}F(u)$, $\downarrow (u)(S) = \downarrow S$.

Proposition 2.3. $\downarrow () : \mathbb{P}F \rightarrow \mathbb{D}F$ is order-preserving which is left adjoint to the inclusion $\mathbb{D}F \hookrightarrow \mathbb{P}F$, i.e. $\downarrow S \subset G \Leftrightarrow S \subset G$ for any $S \in \text{Sub}(F)$ and $G \in \text{Dow}(F)$.

Corollary 2.1. Let F be a posheaf and G a subsheaf of F . G is a downsheaf iff $G = \downarrow G$.

3 Complete Posheaves

Let F be a posheaf on a locale X and A a subsheaf of F . A point $x \in F_p$ is called an upper bound of A if for any point $y \in A_p$ we have $y \leq x$. A least upper bound, otherwise known as a *supremum* or even *sup* for A , is an upper bound p for A such that any upper bound r for A we have $p \leq r$. The *supremum* for A , if it exists, is unique. We will write $\bigvee A$ for the *supremum* of A . Dually, we define a lower bound l of A to be an upper bound of A in the opposite posheaf F^{op} , and *infimum* (shortly *inf*) for A to be the largest lower bound of A , i.e. the least upper bound in F^{op} and write it $\bigwedge A$.

Suppose B is a sub-presheaf of a posheaf F , and \hat{B} is the subsheaf of F generated by B . Then it is clear that \hat{B} and B have the same upper bound set and the same lower bound set. If $\alpha : F \rightarrow G$ is an order-preserving morphism and S is a subsheaf of F , we call the sub-presheaf αS of G defined by $\alpha S(u) = \alpha_u(S(u))$ the preimage of S under

α , and the subsheaf $\hat{\alpha}S$ generated by αS the image of S under α . Suppose $\bigvee S$ exists for a subsheaf S of F , we say that α preserves $\bigvee S$ if $\bigvee \hat{\alpha}S$ exists and $\alpha \bigvee S = \bigvee \hat{\alpha}S$ holds. Similarly we define α preserves $\bigwedge S$ if $\bigwedge \hat{\alpha}S$ exists and $\alpha \bigwedge S = \bigwedge \hat{\alpha}S$ holds.

Proposition 3.1. *If $(\alpha \dashv \beta) : F \rightarrow G$ then α preserves any suprema that exist in F . Dually, β preserves any infima that exist in G .*

Proof Let $S \subseteq F$ be a subsheaf of F with $\bigvee S$ exists. It is clear that $\alpha \bigvee S$ is an upper bound of αS since α is order preserving. Suppose $y \in G_p$ is another upper bound of αS . Then for any point x in S , $\alpha x \leq y$, hence $x \leq \beta y$ by adjointness. Thus we have $\bigvee S \leq \beta y$ which means $\alpha \bigvee S \leq y$. This shows that $\bigvee \hat{\alpha}S = \alpha \bigvee S$. The left follows from the fact that $(\alpha \dashv \beta) : F \rightarrow G$ implies $(\beta \dashv \alpha) : G^{op} \rightarrow F^{op}$. \square

Definition 3.1. Let F be a posheaf on a locale X . F is called complete if the principle ideals embedding $\downarrow : F \rightarrow \mathbb{D}F$ has a left adjoint.

Lemma 3.1. *Let F be a posheaf on a locale X . If each restriction map $F(u) \rightarrow F(v)$ is surjective and it has a left adjoint, then for each $u, v \in \mathcal{O}(X)$, the following square commutes:*

$$\begin{array}{ccc} F(u \vee v) & \longrightarrow & F(u) \\ l_{vu \vee v} \uparrow & & \uparrow l_{u \wedge vu} \\ F(v) & \longrightarrow & F(u \wedge v) \end{array}$$

where $l_{vu \vee v} : F(v) \rightarrow F(u \vee v)$ and $l_{u \wedge vu} : F(u \wedge v) \rightarrow F(u)$ are the left adjoint of the restriction maps $F(u \vee v) \rightarrow F(v)$ and $F(u) \rightarrow F(u \wedge v)$ respectively.

Proof We note that the above square always commutes for $u \leq v$ or $v \leq u$. Suppose $v \not\leq u$, $u \not\leq v$ and let $x \in F(v)$. We first have $((l_{vu \vee v}(x))|_u)|_{u \wedge v} = ((l_{vu \vee v}(x))|_v)|_{u \wedge v} = x|_{u \wedge v}$. Suppose $y \in F(u)$ and $y|_{u \wedge v} = x|_{u \wedge v}$, we have an unique element $z \in F(u \vee v)$ such that $z|_v = x, z|_u = y$. Thus $l_{vu \vee v}(x) \leq z$ and $(l_{vu \vee v}(x))|_u \leq z|_u = y$. This shows that $(l_{vu \vee v}(x))|_u$ is the least element t in $F(u)$ such that $t|_{u \wedge v} = x|_{u \wedge v}$. Hence $l_{u \wedge vu}(x|_{u \wedge v}) = (l_{vu \vee v}(x))|_u$. \square

Proposition 3.2. *Let F be a posheaf on a locale X . The following are equivalent:*

- (1) F is complete.
- (2) For every downsheaf S of F , $\bigvee S$ exists and it can be extended to a global point p of F such that for each $u \in \mathcal{O}(X)$, $p|_u$ is the least element in $F(u)$ satisfying $S^u \subseteq \downarrow p|_u$.
- (3) For every subsheaf S of F , $\bigvee S$ exists and it can be extended to a global point p of F such that for each $u \in \mathcal{O}(X)$, $p|_u$ is the least element in $F(u)$ satisfying $S^u \subseteq \downarrow p|_u$.
- (4) Every $F(u)$ is a complete lattice, and each restriction map $F(u) \rightarrow F(v)$ is surjective and it has both a left adjoint and a right adjoint.

Proof (1) \Leftrightarrow (2) The statement that the principle ideals embedding $\downarrow: F \rightarrow \mathbb{D}F$ has a left adjoint is equivalent to say that we have a morphism $sup: \mathbb{D}F \rightarrow F$ such that for any downsheaf S of F , $sup_u(S^u)$ is the least element in $F(u)$ satisfying $S^u \subseteq \downarrow sup_u(S^u)$ for $u \in \mathcal{O}(X)$. This is equivalent to the condition that for any downsheaf S of F , $\bigvee S$ exists and it can be extended to a global point $sup_{1_X}(S) \in F(1_X)$ such that for any $v \in \mathcal{O}(X)$, $sup_{1_X}(S)|_v$ is the least upper bound of S^v , i.e. the following square commutes

$$\begin{array}{ccc} Dow(F) & \xrightarrow{sup_{1_X}} & F(1_X) \\ \downarrow & & \downarrow \\ Dow(F^v) & \xrightarrow{sup_v} & F(v) \end{array}$$

(3) \Rightarrow (2) is clear. To show (2) \Rightarrow (3), it suffice to show that for any subsheaf S of F , $\bigvee \downarrow S$ exists implies that $\bigvee S$ exists and $\bigvee S = \bigvee \downarrow S$. But it is clear since $\downarrow S$ and S has same upper bound set.

(1) \Rightarrow (4) Suppose $x \in F(u)$, we regard x as a point of F and thus a subsheaf of F . Applied x to the above commute square we have an element $sup_{1_X}(x) \in F(1_X)$ such that $sup_{1_X}(x)|_u = x$. Hence the restriction map $F(1_X) \rightarrow F(u)$ is surjective. If F is a complete posheaf, then it has a largest point- the *supremum* of F . This implies that each $F(u)$ has a largest element, and the restriction maps preserve the top elements. F also has a least point- the *supremum* of the least subsheaf of F . This means that each $F(u)$ has a bottom element and the restriction maps preserve the bottom elements. Suppose $A \subseteq F(u)$ for an $u \in \mathcal{O}(X)$, consider the presheaf \bar{A} determined by A :

$$\bar{A}(u) = \begin{cases} \{x \in F(v) \mid \exists y \in A, x = y|_v\}, & v \leq u \\ \emptyset, & v \not\leq u \end{cases}$$

Write \hat{A} the subsheaf generated by \bar{A} , then $\bigvee \hat{A}$ exists. It is clear that $\bigvee \hat{A}$ is also the least upper bound of A in the poset $F(u)$. Hence $F(u)$ is complete. For $v \leq u$ in $\mathcal{O}(X)$, the commutative of the square

$$\begin{array}{ccc} \mathbb{D}F(u) & \xrightarrow{sup_u} & F(u) \\ \downarrow & & \downarrow \\ \mathbb{D}F(v) & \xrightarrow{sup_v} & F(v) \end{array}$$

implies that $\bigvee \hat{A}|_v = \bigvee \hat{A}^v$, i.e. $(\bigvee A)|_v = \bigvee (A|_v)$. Hence the restriction map $F(u) \rightarrow F(v)$ preserves joins, so has a right adjoint. To show the restriction map $F(u) \rightarrow F(v)$ has a left adjoint, we note that for any element $x \in F(v)$, if we regard x as a point and thus a subsheaf of F , there exists a least element $y \in F(u)$ such that $x \leq y|_v$, i.e. $\min\{y \in F(u) \mid x \leq y|_v\}$ exists. Hence the restriction map $F(u) \rightarrow F(v)$ has a left adjoint.

(4) \Rightarrow (3) Suppose S is a subsheaf of F , write $u = \bigvee \{v \in \mathcal{O}(X) \mid S(v) \neq 0\}$. For $v \leq u$ in $\mathcal{O}(X)$, denote $l_{vu} : F(v) \rightarrow F(u)$ the left adjoint of the restriction map $F(u) \rightarrow F(v)$. Let $s = \bigvee \{l_{vu}(\bigvee S(v)) \mid v \leq u\}$, then it is clear that s is the least upper bound of S . Write $\bar{s} = l_{u1_X}(s)$, then $\bar{s}|_u = s$. We first show $\bar{s}|_v = s|_v = \bigvee S^v$ for $v \leq u$.

In fact $s|_v = (\bigvee \{(l_{wu}(\bigvee S(w)))|_v \mid w \leq v\}) \vee (\bigvee \{(l_{wu}(\bigvee S(w)))|_v \mid w \leq u, w \not\leq v\}) = (\bigvee \{(l_{wv}(\bigvee S(w)))|_v \mid w \leq v\}) \vee (\bigvee \{(l_{wu}(\bigvee S(w)))|_v \mid w \leq u, w \not\leq v\}) = (\bigvee S^v) \vee (\bigvee \{(l_{wu}(\bigvee S(w)))|_v \mid w \leq u, w \not\leq v\})$. For each $w \leq u, w \not\leq v$, by lemma 3.1, we have $(l_{wu}(\bigvee S(w)))|_v = (l_{wv \vee w}(\bigvee S(w)))|_v = l_{v \wedge wv}((\bigvee S(w))|_{v \wedge w}) \leq l_{v \wedge wv}(\bigvee S(v \wedge w))$ since every restriction map preserves sups. Hence $\bigvee \{(l_{wu}(\bigvee S(w)))|_v \mid w \leq u, w \not\leq v\} \leq \bigvee S^v$. This shows $s|_v = \bigvee S^v$.

Suppose $v \not\leq u$. By lemma 3.1 and the above result, we have $\bigvee S^v = l_{u \wedge vv}(\bigvee S^{u \wedge v}) = l_{u \wedge vv}(s|_{u \wedge v}) = (l_{uu \vee v}(s))|_v = \bar{s}|_v$. \square

Example 3.1. Let F be a sheaf on locale X . Consider the power sheaf $\mathbb{P}F$ of F . For any $u \in \mathcal{O}(X)$, $\mathbb{P}F(u) = \text{sub}(F^u)$ is a complete lattice. Moreover, the left adjoint $l_{vu} : F(v) \rightarrow F(u)$ of the restriction map $F(u) \rightarrow F(v)$ sends each subsheaf S of F^v to a subsheaf \check{S} of F^u defined by for any $w \leq u$,

$$\check{S}(w) = \begin{cases} S(w), & w \leq v \\ \emptyset, & w \not\leq v \end{cases}$$

The right adjoint of the restriction map $F(u) \rightarrow F(v)$ sends each subsheaf S of F^v to a subsheaf \hat{S} of F^u generated by the presheaf \bar{S} with $\bar{S}(w) = \{x \in F(w) \mid x|_{w \wedge v} \in S(w \wedge v)\}$, $w \leq u$. Hence $\mathbb{P}F$ is a complete partially ordered sheaf. Similarly, we can show the down-powersheaf $\mathbb{D}F$ of F is a complete partially ordered sheaf.

Example 3.2. Let X be a topological space. Consider the sheaf LSC_X of lower semicontinuous functions into the unit interval $[0, 1]$ on X . Then each $LSC_X(u)$ is a complete lattice, and each restriction map $LSC_X(u) \rightarrow LSC_X(v)$ preserves joins since joins are pointwise. For opens $v \leq u$, and a lower semicontinuous map $f : v \rightarrow [0, 1]$, define $\bar{f} : u \rightarrow [0, 1]$

$$\bar{f}(x) = \begin{cases} f(x), & x \in v \\ 0, & x \in u \setminus v \end{cases}$$

Then \bar{f} is lower semicontinuous and the corresponding $f \mapsto \bar{f}$ forms a left adjoint of the restriction map $LSC_X(u) \rightarrow LSC_X(v)$. Hence LSC_X is a complete partially ordered sheaf.

Corollary 3.1. Let F be a posheaf on a locale X . F is complete if and only if F^{op} is complete.

We write SCL for the category of all complete lattices and surjective maps preserving arbitrary sups and arbitrary infs.

Corollary 3.2. *Let X be a locale. F is a complete posheaf on X if and only if F is a sheaf over SCL satisfying the condition (POS3).*

Let F and G be complete posheaves on a locale X and let $\alpha : F \rightarrow G$ be an order-preserving morphism. Then we have an order preserving morphism $\alpha_* : \mathbb{P}F \rightarrow \mathbb{P}G$ such that for each $u \in \mathcal{O}(X)$ and $S \in \text{sub}(F^u)$, $\alpha_{*u} : \text{sub}(F^u) \rightarrow \text{sub}(G^u)$ maps S to the image of S under α . We call α an sup-preserving morphism if the following square commutes:

$$\begin{array}{ccc} \mathbb{P}F & \xrightarrow{\alpha_*} & \mathbb{P}G \\ \text{sup}_F \downarrow & & \downarrow \text{sup}_G \\ F & \xrightarrow{\alpha} & G \end{array}$$

where $\text{sup}_F : \mathbb{P}(F) \rightarrow F$ and $\text{sup}_G : \mathbb{P}(G) \rightarrow G$ are the left adjoint of the principle ideal embedding $\mathbb{P}F \rightarrow F$ and $\mathbb{P}G \rightarrow G$ respectively.

Proposition 3.3. *Let F and G be complete posheaves on a locale X and $\alpha : F \rightarrow G$ be an order-preserving morphism. The following conditions are equivalent:*

- (1) $\alpha : F \rightarrow G$ is an sup-preserving morphism.
- (2) For each $u \in \mathcal{O}(X)$, $\alpha_u : F(u) \rightarrow G(u)$ preserves joins and the following square commutes for any $v \leq u$ in $\mathcal{O}(X)$

$$\begin{array}{ccc} F(u) & \xrightarrow{\alpha_u} & G(u) \\ f_{uv} \uparrow & & \uparrow g_{uv} \\ F(v) & \xrightarrow{\alpha_v} & G(v) \end{array}$$

where $f_{uv} : F(v) \rightarrow F(u)$ and $g_{uv} : G(v) \rightarrow G(u)$ are the left adjoint of the restriction maps $F(u) \rightarrow F(v)$ and $G(u) \rightarrow G(v)$ respectively.

- (3) α has a right adjoint, i.e. there exists an adjoint pair $\alpha \vdash \beta : F \rightarrow G$.

Proof (1) \Rightarrow (2) Suppose $u \in \mathcal{O}(X)$ and $A \subseteq F(u)$. Write \hat{A} the subsheaf of F generated by A , $\alpha\hat{A}$ the image of \hat{A} under α . Then $\bigvee \hat{A} = \bigvee A$ and $\bigvee \alpha\hat{A} = \bigvee \alpha A$. Hence we have $\alpha_u(\bigvee A) = \bigvee \alpha_u(A)$ by applying the commutative square of the definition. For each $v \leq u$ in $\mathcal{O}(X)$ and $x \in F(v)$, we regard x as a point hence a shbsheaf of F^u . We have $\text{sup}_{F^u}(x) = f_{uv}(x)$, $\text{sup}_{G^u}(x) = g_{uv}(x)$. Hence $\alpha_u f_{uv}(x) = g_{uv} \alpha_v(x)$ by the definition.

(2) \Rightarrow (1) Suppose $u \in \mathcal{O}(X)$, S is a subsheaf of F^u and $\alpha\hat{S}$ the image of S under α . Write $w = \bigvee \{v \in \mathcal{O}(X) \mid S(v) \neq \emptyset\}$. Then $\bigvee S = \bigvee \{f_{wv}(\bigvee S(v)) \mid v \leq w\}$. Thus $\bigvee \alpha\hat{S} = \bigvee \{g_{wv}(\bigvee \alpha_v(S(v))) \mid v \leq w\} = \bigvee \{g_{wv} \alpha_v(\bigvee S(v)) \mid v \leq w\} = \bigvee \{\alpha_w f_{wv}(\bigvee S(v)) \mid v \leq w\} = \alpha_w(\bigvee \{f_{wv}(\bigvee S(v)) \mid v \leq w\}) = \alpha \bigvee S$. By the completeness of F , we know $\text{sup}_{F^u}(S) = f_{uw}(\bigvee S)$ and $\text{sup}_{G^u}(\alpha\hat{S}) = g_{uw}(\bigvee \alpha\hat{S})$. Hence by the commutative square in (2), we have $\alpha_u \text{sup}_{F^u}(S) = \text{sup}_{G^u}(\alpha\hat{S})$. This shows that α is an sup-preserving morphism.

(2) \Leftrightarrow (3) By lemma 2.6, α has a right adjoint β if and only if $\alpha_u : F(u) \rightarrow G(u)$ has a right adjoint $\beta_u : G(u) \rightarrow F(u)$ for every $u \in \mathcal{O}(X)$ and the following square commutes for any $v \leq u$ in $\mathcal{O}(X)$

$$\begin{array}{ccc} G(u) & \xrightarrow{\beta_u} & F(u) \\ \downarrow & & \downarrow \\ G(v) & \xrightarrow{\beta_v} & F(v) \end{array}$$

It equivalent to that each $\alpha_u : F(u) \rightarrow G(u)$ preserves joins and the following square commutes for any $v \leq u$ in $\mathcal{O}(X)$

$$\begin{array}{ccc} F(u) & \xrightarrow{\alpha_u} & G(u) \\ f_{uv} \uparrow & & \uparrow g_{uv} \\ F(v) & \xrightarrow{\alpha_v} & G(v) \end{array}$$

by the uniqueness of adjoint. \square

Example 3.3. Let F and G be sheaves on a locale X , and $\alpha : F \rightarrow G$ be a morphism. Consider the image morphism $\alpha_* : \mathbb{P}F \rightarrow \mathbb{P}G$. For each $u \in \mathcal{O}(X)$, $\alpha_*(u) : \mathbb{P}F(u) \rightarrow \mathbb{P}G(u)$ preserves joins of subsheaves. The left adjoint $\mathbb{P}F(v) \rightarrow \mathbb{P}F(u)$ for a restriction map $\mathbb{P}F(u) \rightarrow \mathbb{P}F(v)$ sends each $S \in F^v$ to its minimal extension $\tilde{S} \in F^u$ defined by

$$\tilde{S}(w) = \begin{cases} S(w), & w \leq v \\ \emptyset, & w \not\leq v \end{cases}$$

Hence the square in proposition 3.3 commutes. This shows that $\alpha_* : \mathbb{P}F \rightarrow \mathbb{P}G$ is an sup-preserving morphism.

Now we consider the finite completeness of posheaves.

Definition 3.2. Let F be a posheaf on X . F is said to be finite sup-complete if $F \rightarrow 1$ and the diagonal $F \rightarrow F \times F$ both has a left adjoint.

Proposition 3.4. Let F be a posheaf on X . The followings are equivalent:

- (1) F is finite sup-complete.
- (2) For every $u \in \mathcal{O}(X)$, $F(u)$ is an sup-semilattice and every restriction map $F(u) \rightarrow F(v)$ preserves finite joins.

Dually, we define a posheaf F to be finite inf-complete if and only if $F \rightarrow 1$ and the diagonal $F \rightarrow F \times F$ both has a right adjoint.

Proposition 3.5. Let F be a posheaf on X . The followings are equivalent:

- (1) F is finite inf-complete.
- (2) For every $u \in \mathcal{O}(X)$, $F(u)$ is a inf-semilattice and every restriction map $F(u) \rightarrow F(v)$ preserves finite meets.

Definition 3.3. Let F be a posheaf on X . F is said to be finite complete if it is both finite sup-complete and finite inf-complete

Proposition 3.6. Let F be a posheaf on X . The followings are equivalent:

- (1) F is finite complete.
- (2) For every $u \in \mathcal{O}(X)$, $F(u)$ is a lattice and every restriction map $F(u) \rightarrow F(v)$ preserves finite joins and finite meets.

By proposition 3.2, we know that every complete posheaf is finite complete.

Now we consider the generalization of another very important class of order algebras-complete Heyting algebras (or frames). For every complete posheaf F , we define a meet morphism $\mu_F : F \times \mathbb{P}F \rightarrow \mathbb{P}F$ of points with subsheaves such that for each $u \in \mathcal{O}(X)$, $x \in F(u)$, and $S \in \text{sub}(F^u)$, $\mu_F(u)(x, S)$ be the subsheaf of F^u generated by the subpresheaf \bar{S} of F^u with $\bar{S}(v) = \{x|_v \wedge y \mid y \in S(v)\}$ for each $v \leq u$. Note that since each restriction map preserves meets so \bar{S} is indeed a sub-presheaf of F^u .

Definition 3.4. Let F be a complete posheaf on a locale X . F is said to be a complete Heyting sheaf (or frame sheaf) if the following square commutes:

$$\begin{array}{ccc} F \times \mathbb{P}F & \xrightarrow{1_F \times \text{sup}_F} & F \times F \\ \mu_F \downarrow & & \downarrow m_F \\ \mathbb{P}F & \xrightarrow{\text{sup}_F} & F \end{array}$$

where $m_F : F \times F \rightarrow F$ is the right adjoint of the diagonal $F \rightarrow F \times F$.

Proposition 3.7. Let F be a complete posheaf on a locale X . The following conditions are equivalent:

- (1) F is a complete Heyting sheaf.
- (2) Each $F(u)$ is a complete Heyting algebra for $u \in \mathcal{O}(X)$, and if $v \leq u$ in $\mathcal{O}(X)$, then $x \wedge l_{uv}(y) = l_{uv}(x|_v \wedge y)$ holds for any $x \in F(u), y \in F(v)$ where $l_{uv} : F(v) \rightarrow F(u)$ is the left adjoint of the restriction map $F(u) \rightarrow F(v)$.

Proof (1) \Rightarrow (2) Suppose $x \in F(u), S \subseteq F(u), u \in \mathcal{O}(X)$. Write \hat{S} for the subsheaf of F^u generated by S . Then we have $\bigvee \hat{S} = \bigvee S$, and $\bigvee \mu_F(u)(x, S) = \bigvee \{x \wedge s \mid s \in S\}$. By the definition, we have $x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$. For $v \leq u$ in $\mathcal{O}(X)$, let $x \in F(u), y \in F(v)$. Regard y as a point and thus a subsheaf of F^u , we have $\text{sup}_F(u)(y) = l_{uv}(y)$ and $\text{sup}_F(u)(\mu_F(u)(x, S)) = l_{uv}(x|_v \wedge y)$. Hence $x \wedge l_{uv}(y) = l_{uv}(x|_v \wedge y)$ by the definition.

(2) \Rightarrow (1) Suppose $x \in F(u)$ and $S \in \text{sub}(F^u)$ a subsheaf of F^u , $u \in \mathcal{O}(X)$. Write $w = \bigvee \{v \in \mathcal{O}(X) \mid S(v) \neq \emptyset\}$. Then $\bigvee S = \bigvee \{l_{uv}(\bigvee S(v)) \mid v \leq w\}$, and $\text{sup}_F(u)(S) = l_{uw}(\bigvee S)$. Thus $\text{sup}_F(u)(\mu_F(u)(x, S)) = l_{uw}(\bigvee \mu_F(u)(x, S)) = l_{uw}(\bigvee \{l_{uv}(\bigvee \{x|_v \wedge y \mid y \in S(v)\} \mid v \leq w\}) = l_{uw}(\bigvee \{l_{uv}(x|_v \wedge \bigvee S(v)) \mid v \leq$

$$w\} = l_{uw}(\bigvee\{x|_w \wedge l_{wv}(\bigvee S(v)) \mid v \leq w\} = l_{uw}(x|_w \wedge \bigvee\{l_{wv}(S(v)) \mid v \leq w\}) = x \wedge l_{uw}(\bigvee\{l_{wv}(S(v)) \mid v \leq w\}) = x \wedge \sup_F(u)(S). \quad \square$$

Example 3.4. Every power sheaf $\mathbb{P}F$ of F is a complete Heyting sheaf. Moreover, every down-powersheaf $\mathbb{D}F$ of F is a complete Heyting sheaf.

Let F and G be two finite inf-complete posheaves on X , we call a morphism $\alpha : F \rightarrow G$ of sheaves preserving finite meets if the following square commutes:

$$\begin{array}{ccc} F \times F & \xrightarrow{\alpha \times \alpha} & G \times G \\ m_F \downarrow & & \downarrow m_G \\ F & \xrightarrow{\alpha} & G \end{array}$$

where $m_F : F \times F \rightarrow F$ and $m_G : G \times G \rightarrow G$ represent the right adjoint of the diagonals $F \rightarrow F \times F$ and $G \rightarrow G \times G$ respectively.

Lemma 3.2. $\alpha : F \rightarrow G$ preserving finite meets if and only if for each $u \in \mathcal{O}(X)$, $\alpha_u : F(u) \rightarrow G(u)$ preserves finite meets.

Definition 3.5. Let F and G be two frame sheaves on X . A morphism $\alpha : F \rightarrow G$ of sheaves is said to be a frame morphism if α is a sup-preserving morphism which also preserves finite meets.

Lemma 3.3. Let F and G be two frame sheaves on X and $\alpha : F \rightarrow G$ is an order-preserving morphism. The following are equivalent:

- (1) α is a frame morphism.
- (2) For each $u \in \mathcal{O}(X)$, $\alpha_u : F(u) \rightarrow G(u)$ is a frame homomorphism and the following square commutes for any $v \leq u$ in $\mathcal{O}(X)$

$$\begin{array}{ccc} F(u) & \xrightarrow{\alpha_u} & G(u) \\ f_{uv} \uparrow & & \uparrow g_{uv} \\ F(v) & \xrightarrow{\alpha_v} & G(v) \end{array}$$

where $f_{uv} : F(v) \rightarrow F(u)$ and $g_{uv} : G(v) \rightarrow G(u)$ are the left adjoint of the restriction maps $F(u) \rightarrow F(v)$ and $G(u) \rightarrow G(v)$ respectively.

We write $\text{FrmSh}(X)$ for the category of all frame sheaves on X and frame morphisms. We now show that the frame sheaves category $\text{FrmSh}(X)$ is equivalent to the category $\mathcal{O}(X)/\text{Frm}$ of frames under $\mathcal{O}(X)$, this shows that the frame sheaves on X are just the internal frames in the localic topos $\text{Sh}(X)$.

Let $f : \mathcal{O}(X) \rightarrow L$ be a frame homomorphism. We define a sheaf $\Phi(F)$ on X such that $\Phi(f)(u) = \{x \in L \mid x \leq f(u)\}$, and for $v \leq u$ in $\mathcal{O}(X)$, the restriction map $\Phi(F)(u) \rightarrow \Phi(F)(v)$ sends an element $x \in \Phi(F)(u)$ to an element $x \wedge f(v) \in \Phi(F)(v)$.

It is clear that $\Phi(f)$ is a frame sheaf. Suppose we are given a commutative diagram of frame homomorphisms

$$\begin{array}{ccc} & \mathcal{O}(X) & \\ f \swarrow & & \searrow g \\ L & \xrightarrow{h} & M \end{array}$$

Then for any $u \in \mathcal{O}(X)$, h can be restricted to a frame homomorphism $\Phi(h)(u) : \Phi(f)(u) \rightarrow \Phi(g)(u)$, and for $v \leq u$, the following square commutes

$$\begin{array}{ccc} \Phi(f)(u) & \xrightarrow{\Phi(h)(u)} & \Phi(g)(u) \\ \uparrow & & \uparrow \\ \Phi(f)(v) & \xrightarrow{\Phi(h)(v)} & \Phi(g)(v) \end{array}$$

Hence $\Phi(f) : \Phi(f) \rightarrow \Phi(g)$ is a frame morphism between frame sheaves. This shows that $\Phi : \mathcal{O}(X)/Frm \rightarrow FrmSh(X)$ is a functor.

Theorem 3.1. *The frame sheaves category $FrmSh(X)$ is equivalent to the category $\mathcal{O}(X)/Frm$ of frames under $\mathcal{O}(X)$.*

Proof Suppose F is a frame sheaf on X , write $\Psi(F) = F(1_X)$ where 1_X be the largest element of $\mathcal{O}(X)$. For $u \in \mathcal{O}(X)$, let $f : \mathcal{O}(X) \rightarrow \Psi(F)$, $u \mapsto l_{u1_X}(1_{F(u)})$ where $l_{u1_X} : F(u) \rightarrow F(1_X)$ be the left adjoint of the restriction map $F(1_X) \rightarrow F(u)$ and $1_{F(u)}$ be the largest element of $F(u)$. We first show that $f : \mathcal{O}(X) \rightarrow \Psi(F)$ is a frame homomorphism.

Let $u, v \in \mathcal{O}(X)$. We have $l_{u \wedge v u}(1_{F(u \wedge v)}) = (l_{vu \vee v}(1_{F(v)}))|_u$ by lemma 3.1, it implies that $l_{u \wedge v u}(1_{F(u \wedge v)}) = (l_{v1_X}(1_{F(v)}))|_u$. This implies that $l_{u \wedge v 1_X}(1_{F(u \wedge v)}) = l_{u1_X}((l_{v1_X}(1_{F(v)}))|_u) = l_{u1_X}((l_{v1_X}(1_{F(v)}))|_u \wedge 1_{F(u)}) = l_{v1_X}(1_{F(v)}) \wedge l_{u1_X}(1_{F(u)})$ by proposition 3.7. Hence f preserves finite meets. Suppose $\{u_i \mid i \in I\} \subseteq \mathcal{O}(X)$, write $t = \bigvee l_{u_i \vee u_i}(1_{F(u_i)})$. Then $t|_{u_i} \geq (l_{u_i \vee u_i}(1_{F(u_i)}))|_{u_i} = 1_{F(u_i)}$. Thus $t = 1_{F(\bigvee u_i)}$ by sheaf axiom. This implies that $l_{\bigvee u_i 1_X}(1_{F(\bigvee u_i)}) = l_{\bigvee u_i 1_X}(\bigvee l_{u_i \vee u_i}(1_{F(u_i)})) = \bigvee l_{u_i 1_X}(1_{F(u_i)})$ since $l_{\bigvee u_i 1_X}$ preserves joins.

Suppose F and G are frame sheaves on X and $\alpha : F \rightarrow G$ is a frame morphism. Write $\Psi(\alpha) = \alpha_{1_X} : \Psi(F) = F(1_X) \rightarrow \Psi(G) = G(1_X)$. The above argument shows that $\Psi : FrmSh(X) \rightarrow \mathcal{O}(X)/Frm$ is a functor.

We now show that $\Phi\Psi$ is isomorphic to the identity on $FrmSh(X)$ and $\Psi\Phi$ is isomorphic to the identity on $\mathcal{O}(X)/Frm$.

Suppose F is a frame sheaf on X . For $v \leq u$ in $\mathcal{O}(X)$, $l_{vu} : F(v) \rightarrow F(u)$ is the left adjoint of restriction map $F(u) \rightarrow F(v)$. We have $l_{vu}(y \wedge z) = l_{vu}((l_{vu}(y))|_v \wedge z) = l_{vu}(y) \wedge l_{vu}(z)$ by proposition 3.7. Thus $l_{vu} : F(v) \rightarrow F(u)$ is a frame mono-homomorphism, in particular, $l_{u1_X} : F(u) \rightarrow F(1_X)$ is a frame mono-homomorphism for any $u \in \mathcal{O}(X)$. Moreover, if $x \leq l_{u1_X}(1_{F(u)})$, then $x = x \wedge l_{u1_X}(1_{F(u)}) = l_{u1_X}(x|_u \wedge$

$1_{F(u)} = l_{u1_X}(x|_u)$. This shows that $l_{u1_X} : F(u) \rightarrow \Phi\Psi(F)(u) = \downarrow l_{u1_X}(1_{F(u)})$ is an isomorphism of frames. For naturality of the isomorphism, we note that $l_{u1_X}(x) \wedge l_{v1_X}(1_{F(v)}) = l_{v1_X}(l_{u1_X}(x)|_v \wedge 1_{F(u)}) = l_{v1_X}(x|_v)$. Hence $\Phi\Psi$ is isomorphic to the identity on $\text{FrmSh}(X)$. It is clear that $\Psi\Phi$ is isomorphic to the identity on $\mathcal{O}(X)/\text{Frm}$. \square

4 Partially Ordered Sheaf Locales

It is well known that the category $\text{Sh}(X)$ of sheaves on a locale X is equivalent to the slice category LH/X of locales and local homeomorphisms over X (see Johnstone [5]). In this section, we first give an explicit description of the construction of the associated sheaf locales and show directly that the category $\text{Sh}(X)$ of sheaves on a locale X is equivalent to the slice category LH/X of locales and local homeomorphisms over X , then we give characterizations of partially ordered sheaf locales and complete partially ordered sheaf locales respectively.

Recall a localic map $f : X \rightarrow Y$ is said to be local homeomorphism if X can be covered by open sublocales U for which the composite $U \rightarrowtail X \rightarrow Y$ is isomorphic to the inclusion of an open sublocale of Y . We will write LH for the category of locales and local homeomorphisms.

Let X be a locale, $P \in [\mathcal{O}(X)^{\text{op}}, \text{Set}]$ be a presheaf on X . For $s_i \in P(u_i), u_i \in \mathcal{O}(X), i = 1, \dots, n$, write $\epsilon_P(s_1, \dots, s_n) = \bigvee \{u \leq u_1 \wedge \dots \wedge u_n \mid s_i|_u = s_j|_u, 1 \leq i, j \leq n\}$. If P is a sheaf, then it is clear that $\epsilon_P(s_1, \dots, s_n)$ is the largest open sublocale u such that $s_i|_u = s_j|_u$ for every $i, j = 1, \dots, n$.

Lemma 4.1. (1) $\epsilon_P(s_1, \dots, s_n, t_1, \dots, t_m) \leq \epsilon_P(s_1, \dots, s_n) \wedge \epsilon_P(t_1, \dots, t_m)$.

(2) $\epsilon_P(s_1, \dots, s_n, t_1, \dots, t_m) = \epsilon_P(s_1, \dots, s_n) \wedge \epsilon_P(t_1, \dots, t_m)$ for any $s_i \in P(u_i), t_j \in P(v_j)$ where $u_i, v_j \in \mathcal{O}(X)$ if and only if P is the terminal object of $[\mathcal{O}(X)^{\text{op}}, \text{Set}]$.

Let X be a locale, $P \in [\mathcal{O}(X)^{\text{op}}, \text{Set}]$. We define $\Lambda(P)$ the frame of all functions $f : \coprod P(u) \rightarrow \mathcal{O}(X)$ with pointwise partial order such that $\forall f \in \Lambda(P)$ satisfying:

- (1 $_{\Lambda}$) $f(s) \leq u$ for $s \in P(u)$;
- (2 $_{\Lambda}$) $f(s) \wedge \epsilon_P(s, t) = f(t) \wedge \epsilon_P(s, t)$

where $\coprod P(u)$ be the disjoint union of all $P(u)$ for $u \in \mathcal{O}(X)$. Equivalently, $\Lambda(P)$ be the subframe of the frame product $\prod_{s \in \coprod P(u)} \downarrow u_s$, where $u_s = u$ for $s \in P(u)$, such that each element (x_s) of $\Lambda(P)$ satisfying $x_s \wedge \epsilon_P(s, t) = x_t \wedge \epsilon_P(s, t)$ for all $s, t \in \coprod P(u)$. Let

$$p^* : \mathcal{O}(X) \rightarrow \Lambda(P), \quad x \mapsto (x \wedge u_s), s \in P(u_s), u_s \in \mathcal{O}(X)$$

then it is clear that p^* is a frame homomorphism.

Lemma 4.2. $p : \Lambda(P) \rightarrow X$ is a local homeomorphism.

Proof. Suppose $s \in P(u)$ and $p_s^* : \Lambda(F) \rightarrow \downarrow u$ be the s 'th projection. It is clear that p_s^* is surjective since $(x \wedge u_s) \in \Lambda(F)$ for any $x \leq u$. Given $(x_t), (x'_t) \in \Lambda(P)$, we have $x'_s \leq x_s \Leftrightarrow x'_t \wedge \epsilon_P(s, t) = x'_s \wedge \epsilon_P(s, t) \leq x_s \wedge \epsilon_P(s, t) = x_t \wedge \epsilon_P(s, t) \leq x_t$ for all $t \in \coprod P(u) \Leftrightarrow x'_t \leq \epsilon_P(s, t) \rightarrow x_t$ for all $t \in \coprod P(u) \Leftrightarrow (x'_t) \leq (\epsilon_P(s, t)) \rightarrow (x_t)$. Hence the nucleus induced by p_s^* is just the open nucleus $(\epsilon_P(s, t)) \rightarrow ()$, that is $\downarrow u$ is isomorphic to the open sublocale $(\epsilon_P(s, t))$ of $\Lambda(P)$. This shows that the set of opens $\{(\epsilon_P(s, t)) \mid s \in \coprod P(u)\}$ of $\Lambda(P)$ form a cover of $\Lambda(P)$ such that each composite $(\epsilon_P(s, t)) \hookrightarrow \Lambda(P) \rightarrow X$ is isomorphic to an open inclusion. \square

Proposition 4.1. $\Lambda : [\mathcal{O}(X)^{op}, Set] \rightarrow LH/X$ is a functor.

Proof. Let $F, G \in [\mathcal{O}(X)^{op}, Set]$ and $\alpha : F \rightarrow G$ be a natural transformation. define $\Lambda(\alpha)^* : \Lambda(G) \rightarrow \Lambda(F)$ as

$$(x_t) \mapsto (x'_s), x'_s = x_t \text{ for } s \in F(u), t \in G(u) \text{ and } \alpha_u(s) = t$$

Note that $\epsilon_F(s_1, s_2) \leq \epsilon_G(t_1, t_2)$ for all $s_1 \in F(u), s_2 \in F(v), t_1 \in G(u), t_2 \in G(v)$ with $t_1 = \alpha_u(s_1), t_2 = \alpha_v(s_2)$ by the natural transformation of α . So $x'_{s_1} \wedge \epsilon_F(s_1, s_2) = x'_{s_2} \wedge \epsilon_F(s_1, s_2)$ for all $s_1, s_2 \in \coprod F(u)$. This shows that $\Lambda(\alpha)^*$ is well defined. It is clear that $\Lambda(\alpha)^*$ is a frame homomorphism such that $\Lambda(\alpha)^* g^* = f^*$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \Lambda(F) & \xrightarrow{\Lambda(\alpha)} & \Lambda(G) \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

If $\alpha : F \rightarrow G, \beta : G \rightarrow H$ are natural transformations, then it is clear $\Lambda(\beta\alpha) = \Lambda(\beta)\Lambda(\alpha)$. Hence $\Lambda : [\mathcal{O}(X)^{op}, Set] \rightarrow LH/X$ is a functor. \square

Let X be a locale. Recall the cross-sections functor $\Gamma : Loc/X \rightarrow Sh(X)$ defined in [5]: given a locale $p : E \rightarrow X$ over X , $\Gamma(p)$ is the sheaf such that $\Gamma(p)(u)$ be the set of all continuous sections of p over u , where $u \in \mathcal{O}(X)$, i.e. localic maps $s : \downarrow u \rightarrow E$ such that the composite ps is the inclusion $\downarrow u \hookrightarrow X$.

Proposition 4.2. For any locale X , $\Lambda : [\mathcal{O}(X)^{op}, Set] \rightarrow Loc/X$ is left adjoint to the functor $\Gamma : Loc/X \rightarrow [\mathcal{O}(X)^{op}, Set]$. The unit $\eta : 1_{[\mathcal{O}(X)^{op}, Set]} \rightarrow \Gamma\Lambda$ is defined as:

$$\eta_{P_u} : P(u) \rightarrow \Gamma\Lambda(P)(u), s \mapsto p_s$$

while the counit $\varepsilon : \Lambda\Gamma \rightarrow 1_{Loc/X}$ is defined such that for any locale $f : Y \rightarrow X$ over X ,

$$\varepsilon_f^* : Y \rightarrow \Lambda\Gamma(f), y \mapsto (s^*(y))$$

Proof. We first show that η and ε are both natural transformations.

Suppose $P \in [\mathcal{O}(X)^{op}, Set]$. Note that for $v \leq u$ in $\mathcal{O}(X)$ and $s \in P(u)$, we have $\epsilon_P(s, s|_v) = v$. Hence $p_s i = p_{s|_v}$ where $i : v \hookrightarrow u$ is the inclusion. This shows $\eta_P :$

$P \rightarrow \Gamma\Lambda(P)$ is a natural transformation. It is readily to verify that $\eta_P : P \rightarrow \Gamma\Lambda(P)$ is natural for P .

For natural transformation $\varepsilon : \Lambda\Gamma \rightarrow 1_{Loc/X}$ it is clear.

Next we observe that η and ε have the property that both composites

$$\Gamma \xrightarrow{\eta^\Gamma} \Gamma\Lambda\Gamma \xrightarrow{\Gamma\varepsilon} \Gamma, \quad \Lambda \xrightarrow{\Lambda\eta} \Lambda\Gamma\Lambda \xrightarrow{\varepsilon\Lambda} \Lambda$$

are identities. For the first composite, given a locale $f : Y \rightarrow X$ over X and $s \in \Gamma(f)(u)$, $\eta_{\Gamma(f)_u}$ sends s to the s 'th projection $p_s : u \rightarrow \Lambda\Gamma(f)$, and then $\Gamma(\varepsilon_f)_u$ sends it to $\varepsilon_f p_s = s$. Similarly, for a presheaf P on X , the second composite first sends $y = (x_t) \in \Lambda(P)$ to $(s^*(y)) \in \Lambda\Gamma\Lambda(P)$ by $\varepsilon_{\Lambda(P)}^*$ and then sends it to $(p_t^*(y)) = (x_t)$ by $\Lambda(\eta_P)^*$. \square

Corollary 4.1. *The functor Λ and Γ in Proposition 5.2 restrict to an equivalence of categories*

$$Sh(X) \rightleftarrows LH/X$$

Moreover, $Sh(X)$ is a reflective subcategory of $[\mathcal{O}(X)^{op}, Set]$, and LH/X is a coreflective subcategory of Loc/X .

Proof. Let $f : Y \rightarrow X$ be a local homeomorphism. Then we have a cover $\bigvee y_i = Y$ such that each composite $\downarrow y_i \xrightarrow{s_i} Y \xrightarrow{f} X$ is an open inclusion. For $y, y' \in \mathcal{O}(Y)$ with $\varepsilon_f^*(y) = \varepsilon_f^*(y')$, we have $y = \bigvee y \wedge y_i = \bigvee s_i^*(y) = \bigvee s_i^*(y') = y'$. Hence ε_f^* is one to one. For each $(x_s) \in \Lambda\Gamma(f)$, there exist $y_s \in \mathcal{O}(Y)$ such that $x_s = s^*(y_s)$ for every $s \in \Gamma(f)$. But $s^*(y_s) \wedge \epsilon(s, t) = t^*(y_t) \wedge \epsilon(s, t) = s|_{\epsilon(s, t)}^*(y)$ for $s, t \in \Gamma(f)$ implies that $y_s = y_t = y$. Thus $(x_s) = (s^*(y))$ for some $y \in \mathcal{O}(Y)$. This shows that $\varepsilon_f^* : Y \rightarrow \Lambda\Gamma(f)$ is an isomorphism.

Suppose we are given a sheaf F on X , u is an open of X and $\lambda : u \rightarrow \Lambda(F)$ be a continuous section over u . Then for any $s \in F(v)$ and $s' \in F(v')$, we have $\lambda^*((\epsilon_F(s, t))) \wedge \lambda^*((\epsilon_F(s', t))) = \lambda^*(((\epsilon_F(s, t)) \wedge (\epsilon_F(s', t)))) \leq \lambda^*((\epsilon_F(s, s'))) \leq \lambda^*((\epsilon_F(s, s')) \wedge u_s) = \epsilon_F(s, s')$, and also $\bigvee_{s \in \bigcup F(v)} \lambda^*((\epsilon_F(s, t))) = 1_u$. Hence $\{s|_{\lambda^*((\epsilon_F(s, t)))} \mid s \in \bigcup F(v)\}$ form a compatible family, and so they patch to a unique element $\bar{s} \in F(u)$ such that $\bar{s}|_{\lambda^*((\epsilon_F(s, t)))} = s|_{\lambda^*((\epsilon_F(s, t)))}$ for any $s \in \bigcup F(v)$. By the pullback property of the following diagram:

$$\begin{array}{ccc} \downarrow \lambda^*((\epsilon_F(s, t))) & \longrightarrow & \downarrow (\epsilon_F(s, t)) \\ \downarrow & & \downarrow \\ \downarrow u & \xrightarrow{\lambda} & \Lambda(F) \end{array}$$

we know that λ must be the \bar{s} 'th projection $p_{\bar{s}}$. Hence $\eta_{P_u} : P(u) \rightarrow \Gamma\Lambda(P)(u)$ is a bijection. \square

Let $f : Y \rightarrow X$ be a locale morphism and F be the corresponding frame sheaf under the equivalence of theorem 3.1. Recall that F is called spatial in the localic topos $Sh(X)$

if F is isomorphic to a subframe of a power object, equivalently, if there exists a locale epimorphism $h : E \rightarrow Y$ such that the composition fh is a local homeomorphism. By proposition 5.2, we know that F is spatial if and only if the counit $\varepsilon : \Lambda\Gamma \rightarrow 1_{Loc/X}$ is an epimorphism. Thus we have the following result.

Corollary 4.2. *Let $f : Y \rightarrow X$ be a locale morphism. $f : Y \rightarrow X$ corresponds to a spatial frame if and only if for $\forall y, z \in \mathcal{O}(Y)$, $y \neq z$, there exists a continuous section s over some open $u \in \mathcal{O}(X)$ such that $s^*(y) \neq s^*(z)$.*

Let $f : Y \rightarrow X$ be a local homeomorphism and $u, v \in \mathcal{O}(X)$ be two opens with $v \leq u$. If $s \in \Gamma(f)(u)$ be a section over u , then we have section $s|_v$ over v such that for any $y \in \mathcal{O}(Y)$, $s|_v^*(y) = s^*(y) \wedge v$. We call it the restriction of s to v . Now we define the concept of partially ordered sheaf locales in such a way as to make the equivalence $Sh(X) \rightleftharpoons LH/X$ in corollary 5.1 true for the category of partially ordered sheaves.

Definition 4.1. Let $f : Y \rightarrow X$ be a local homeomorphism. f is said to be a partially ordered sheaf locale if

(POSL1) For any $u \in \mathcal{O}(X)$, the set $\Gamma(f)(u)$ of all sections on u is a partially ordered set.

(POSL2) If $v \leq u$ in $\mathcal{O}(X)$, $s, t \in \Gamma(f)(u)$ be two section on u with $s \leq t$, then $s|_v \leq t|_v$ in $\Gamma(f)(v)$.

(POSL3) If we have a cover $u = \bigvee u_i$ in $\mathcal{O}(X)$, and two sections $s, t \in \Gamma(f)(u)$ such that $s|_{u_i} \leq t|_{u_i}$ in $\Gamma(f)(u_i)$ for each u_i then $s \leq t$ in $\Gamma(f)(u)$.

Given a map of locales over X with $f : Y \rightarrow X$ and $g : Z \rightarrow X$ are both local homeomorphisms

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ & \searrow f & \swarrow g \\ & Z & \end{array}$$

We call ϕ an order-preserving map if for each $u \in \mathcal{O}(X)$, the map

$$\phi_u : \Gamma(f)(u) \rightarrow \Gamma(g)(u), s \mapsto \phi s$$

is an order-preserving map between posets. We write $POLH/X$ for the category of all partially ordered sheaf locales over X and order-preserving maps, and \mathfrak{POS}_X for the category of all posheaves on X and order-preserving morphisms. Then we have

Corollary 4.3. *The functors Λ and Γ in Proposition 4.2 restrict to an equivalence of categories*

$$\mathfrak{POS}_X \rightleftharpoons POLH/X$$

To characterize complete partially ordered sheaves in terms of sheaf locales, we need to introduce the concepts of complete partially ordered sheaf locales.

Definition 4.2. Let $f : Y \rightarrow X$ be a local homeomorphism. f is said to be a complete partially ordered sheaf locale if

(CPOSL1) For any $u \in \mathcal{O}(X)$, the set $\Gamma(f)(u)$ of all sections on u is a complete lattice.

(CPOSL2) If $v \leq u$ in $\mathcal{O}(X)$, then the restriction map $\Gamma(f)(u) \rightarrow \Gamma(f)(v)$, $s \mapsto s|_v$ is surjective, and preserves arbitrary joins and meets.

(CPOSL3) If we have a cover $u = \bigvee u_i$ in $\mathcal{O}(X)$, and two sections $s, t \in \Gamma(f)(u)$ such that $s|_{u_i} \leq t|_{u_i}$ in $\Gamma(f)(u_i)$ for each u_i then $s \leq t$ in $\Gamma(f)(u)$.

We write $CPOLH/X$ for the category of all complete partially ordered sheaf locales over X and order-preserving maps, \mathfrak{POS}_X for the category of all complete posheaves on X and order-preserving morphisms. Then we have

Corollary 4.4. *The functors Λ and Γ in Proposition 4.2 restrict to an equivalences of categories*

$$\mathfrak{POS}_X \rightleftarrows CPOLH/X$$

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Partially Ordered Sheaves on a Locale. II ^{*}

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Abstract

This paper is a continuation of our paper [1]. In this paper, we first introduce the concept of directed complete partially ordered sheaves (shortly dcpo-sheaves) on a given locale. Some internal characterizations of dcpo-sheaves and meet continuous dcpo-sheaves are given respectively. We also give characterizations of continuous posheaves and completely distributive posheaves, and show that an algebraic completely distributive posheaf is spatial.

Keywords: sheaf; partially ordered sheaf; continuous partially ordered sheaf.

Mathematics Subject Classifications(2000): 18F20; 06A06.

1 Introduction

In classical order algebraical theory, continuous dcpo (or domains) are very important class of posets. It also closely concerned with topology, i.e. a continuous frame is just a sober locally compact topology. In this paper we first introduce the concept of directed complete partially ordered sheaves on a locale X . Some internal characterizations of directed complete partially ordered sheaves and meet continuous partially ordered sheaves are presented respectively. Then we introduce the concept of continuous posheaves which are continuous directed complete partially ordered objects in a localic topos $Sh(X)$. We show that continuous frame sheaves correspond to open frame homomorphisms satisfying an additional condition. We also show that an algebraic completely distributive posheaf is spatial. Throughout this paper, when we write X for a locale, we will write $\mathcal{O}(X)$ for the corresponding frame.

2 Directed complete posheaves

Let X be a locale and F a sheaf on X . Recall that a point of F to be a morphism $p : \hat{1} \rightarrow F$ with $\hat{1}$ a subsheaf of the terminal sheaf 1 . A point of the form $1 \rightarrow F$ will

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be called a global point of F . For a point $p : \hat{1} \rightarrow F$ of F , we write $\text{dom}(p)$ for the largest open $u \in \mathcal{O}(X)$ with $p(u) \neq \emptyset$, i.e. $\text{dom}(p) = \bigvee \{u \in \mathcal{O}(X) \mid p(u) \neq \emptyset\}$, and call it the domain of p . The set of all points of a sheaf F will be denoted by F_p .

Let F be a posheaf on X , and $D \subseteq F_p$ a family of points. We call D a directed family if for any two points p_1, p_2 in D with $\text{dom}(p_1) \leq \text{dom}(p_2)$, there exists a point $p \in D$ such that $p_1 \leq p$ and $p_2 \leq p$.

Definition 2.1. A posheaf F is said to be directed if there exists a directed family $D \subseteq F_p$ which can generate F .

By the above definition, we know that a posheaf F is directed if and only if for $\forall u \in \mathcal{O}(X)$, there exists a directed set $D(u) \subseteq F(u)$ such that for any two elements $x \in D(u), y \in D(v)$ with $u \leq v$, there exists an element $z \in D(w)$ such that $x \leq z|_u, y \leq z|_v$, and for $\forall x \in F(u)$ there is a cover $u = \bigvee u_i$ satisfying $x|_{u_i} \in D(u_i)$. If a directed subsheaf G of F is also a downsheaf then we call it an ideal of F .

For a posheaf F on a locale X , note that for any directed subsheaf D of F , D^u is still directed, hence we can construct a directed-powersheaf $\mathcal{D}F$ of F defined by $\mathcal{D}F(u) = D(F^u)$ where $D(F^u)$ is the set of all directed subsheaves of F^u , and each restriction map $\mathcal{D}F(u) \rightarrow \mathcal{D}F(v)$ is same as the restriction map for the power sheaf $\mathbb{P}F$. So $\mathcal{D}F$ is a subsheaf of $\mathbb{P}F$.

If p is a point of a posheaf F , the principle ideal $\downarrow p$ is clearly a directed subsheaf of F . Hence the principle ideal embedding morphism $\downarrow : F \rightarrow \mathbb{P}F$ can be factored through $\mathcal{D}F$,

$$\downarrow : F \rightarrow \mathcal{D}F$$

Definition 2.2. A posheaf F is called directed complete if the principle ideal embedding $\downarrow : F \rightarrow \mathcal{D}(F)$ has a left adjoint.

Proposition 2.1. For any posheaf F on a locale X , the following are equivalent:

- (1) F is directed complete.
- (2) For every directed subsheaf D of F , $\bigvee D$ exists and it can be extended to a global point p of F such that for each $u \in \mathcal{O}(X)$, $p|_u$ is the least element in $F(u)$ satisfying $D^u \subseteq \downarrow p|_u$.
- (3) For every ideal D of F , $\bigvee D$ exists and it can be extended to a global point p of F such that for each $u \in \mathcal{O}(X)$, $p|_u$ is the least element in $F(u)$ satisfying $D^u \subseteq \downarrow p|_u$.
- (4) Every restriction map is surjective. For every $u \in \mathcal{O}(X)$, $F(u)$ is a directed complete poset. Every restriction map $F(v) \rightarrow F(u)$ preserves directed joins and has a left adjoint.

Proof (1) \Leftrightarrow (2) is same as the proof of Proposition 3.2 in [1].

(2) \Rightarrow (3) is clear.

(3) \Rightarrow (2) Suppose D is a directed subsheaf of F , then $\downarrow D$ is also directed hence an ideal and $\bigvee D = \bigvee \downarrow D$. By (3) $\bigvee D$ can be extended to a global point p of F such that for each $u \in \mathcal{O}(X)$, $p|_u$ is the least element in $F(u)$ satisfying $(\downarrow D)^u \subseteq \downarrow p|_u$. We note that $(\downarrow D)^u = \downarrow D^u$, hence (2) holds.

(1) \Rightarrow (4) To show each restriction map $F(1_X) \rightarrow F(u)$ is surjective, we only need to note that for any $x \in F(u)$, regarded as a point of F , it is directed. Suppose $D \subseteq F(u)$ is directed. Denote \bar{D} the presheaf generated by D :

$$\bar{D}(v) = \begin{cases} \{x \in F(v) \mid \exists y \in D, x = y|_v\}, & v \leq u \\ \emptyset, & v \not\leq u \end{cases}$$

Write \hat{D} the sheaf generated by \bar{D} , then \hat{D} is directed and $\bigvee \hat{D}$ exists. Similar to the proof of proposition 3.2 in [1], we can show that $\bigvee \hat{D}$ is also the least upper bound of D in the poset $F(u)$, and for $v \leq u$, we have $(\bigvee D)|_v = \bigvee D|_v$. Hence $F(u)$ is directed complete and the restriction map $F(u) \rightarrow F(v)$ preserves directed joins. Moreover, $F(u) \rightarrow F(v)$ has a left adjoint.

(4) \Rightarrow (2) Suppose A is a directed subsheaf of F , and $D \subseteq F_p$ is a directed family which can generate A . Regard D as a sub-presheaf of F , we know D and A have the same upper bound set. For every $u \in \mathcal{O}(X)$, write $D_u = \{x \in F(u) \mid \exists p \in D, x = p|_u\}$. Then D_u is directed in $F(u)$ hence $\bigvee D_u$ exists. For $v \leq u$ in $\mathcal{O}(X)$, $(\bigvee D_u)|_v = \bigvee D_u|_v \leq \bigvee D_v$ since the restriction map $F(u) \rightarrow F(v)$ preserves directed joins. For any $x \in D_v$, by the directness of D there exists $y \in D_u$ such that $x \leq y|_v$, hence the converse inequality $\bigvee D_v \leq (\bigvee D_u)|_v$ holds. This shows that $\{\bigvee D_u \mid D_u \neq \emptyset\}$ is a compatible family, hence it can be patched to a unique element $p \in F(\bigvee \{u \in \mathcal{O}(X) \mid D_u \neq \emptyset\})$. It is clear that p is the least upper bound of A . The point p can be extended to a global point \bar{p} of F such that for each $u \in \mathcal{O}(X)$, $\bar{p}|_u$ is the least element in $F(u)$ satisfying $D^u \subseteq \downarrow \bar{p}|_u$ since the restriction map $F(1_X) \rightarrow F(\bigvee \{u \in \mathcal{O}(X) \mid D_u \neq \emptyset\})$ has a left adjoint. \square

Corollary 2.1. *Let X be a locale and F a posheaf on X . The following conditions are equivalent:*

- (1) *F is complete.*
- (2) *F is finite upper-complete and directed complete.*

We write $DCPO^*$ for the category of all directed complete partially ordered sets and surjective maps having left adjoint and preserving directed joins. Then we have

Corollary 2.2. *Let X be a locale. F is a directed complete posheaf on X if and only if F is a sheaf over $DCPO^*$ and satisfying the condition (POS3).*

Now we consider those morphisms which preserve directed sups between directed complete posheaves. suppose $\alpha : F \rightarrow G$ is an order-preserving morphism and S is a

directed subsheaf of F . If $D \subseteq F_p$ is a directed family which generates S , then the image αD of D under α is a directed family which generates the image $\hat{\alpha}S$ of S , hence $\hat{\alpha}S$ is directed. Let F and G be two directed complete posheaves on a locale X and $\alpha : F \rightarrow G$ be an order-preserving morphism. Since the image of a directed subsheaf is directed, we can construct an order-preserving morphism $\alpha_* : \mathcal{D}F \rightarrow \mathcal{D}G$ such that for each $u \in \mathcal{O}(X)$ and $S \in \mathcal{D}(F^u)$, $\alpha_{*u} : \mathcal{D}(F^u) \rightarrow \mathcal{D}(G^u)$ maps S to the image of S under α .

Definition 2.3. Let F and G be two directed complete posheaves on a locale X and $\alpha : F \rightarrow G$ be an order-preserving morphism. We call α preserving directed sups if the following square commutes:

$$\begin{array}{ccc} \mathcal{D}F & \xrightarrow{\alpha_*} & \mathcal{D}G \\ \text{sup}_F \downarrow & & \downarrow \text{sup}_G \\ F & \xrightarrow{\alpha} & G \end{array}$$

where $\text{sup}_F : \mathcal{D}(F) \rightarrow F$ and $\text{sup}_G : \mathcal{D}(G) \rightarrow G$ are respectively the left adjoint of the principle ideal embedding $F \rightarrow \mathcal{D}F$ and $G \rightarrow \mathcal{D}G$.

Similar to the proof of proposition 3.3 in [1], we have the following result.

Proposition 2.2. *Let F and G be directed complete posheaves on a locale X and let $\alpha : F \rightarrow G$ be an order-preserving morphism. The following conditions are equivalent:*

- (1) $\alpha : F \rightarrow G$ preserves directed sups.
- (2) For each $u \in \mathcal{O}(X)$, $\alpha_u : F(u) \rightarrow G(u)$ preserves directed joins and the following square commutes for any $v \leq u$ in $\mathcal{O}(X)$

$$\begin{array}{ccc} F(u) & \xrightarrow{\alpha_u} & G(u) \\ f_{uv} \uparrow & & \uparrow g_{uv} \\ F(v) & \xrightarrow{\alpha_v} & G(v) \end{array}$$

where $f_{uv} : F(v) \rightarrow F(u)$ and $g_{uv} : G(v) \rightarrow G(u)$ are the left adjoint of the restriction maps $F(u) \rightarrow F(v)$ and $G(u) \rightarrow G(v)$ respectively.

Now we consider the generalization of meet continuous semilattices. Recall that a posheaf F is said to be finite inf-complete if $F \rightarrow 1$ and the diagonal $F \rightarrow F \times F$ both has a right adjoint. An finite inf-complete posheaf F can be characterized as a posheaf F such that for every $u \in \mathcal{O}(X)$, $F(u)$ is a inf-semilattice and every restriction map $F(u) \rightarrow F(v)$ preserves finite meets. Let F be a directed complete finite inf-complete posheaf. We define a meet morphism $\mu_F : F \times \mathbb{P}F \rightarrow \mathbb{P}F$ of points with subsheaves such that for each $u \in \mathcal{O}(X)$, $x \in F(u)$, and $S \in \text{sub}(F^u)$, $\mu_F(u)(x, S)$ be the directed subsheaf of F^u generated by the sub-presheaf \bar{S} of F^u with $\bar{S}(v) = \{x|_v \wedge y \mid y \in S(v)\}$ for each $v \leq u$. Note that since each restriction map preserves meets so \bar{S} is indeed a sub-presheaf of F^u .

Definition 2.4. Let F be a directed complete finite inf-complete posheaf on a locale X . F is said to be a meet continuous posheaf if the following square commutes:

$$\begin{array}{ccc} F \times \mathbb{P}F & \xrightarrow{1_F \times \text{sup}_F} & F \times F \\ \mu_F \downarrow & & \downarrow m_F \\ \mathbb{P}F & \xrightarrow{\text{sup}_F} & F \end{array}$$

where $m_F : F \times F \rightarrow F$ is the right adjoint of the diagonal $F \rightarrow F \times F$.

Similar to the proof of Proposition 3.7 in [1], we have the following result.

Proposition 2.3. *Let F be a directed complete finite inf-complete posheaf on a locale X . The following conditions are equivalent:*

- (1) F is a meet continuous posheaf.
- (2) Each $F(u)$ is a meet continuous semilattice for $u \in \mathcal{O}(X)$, and if $v \leq u$ in $\mathcal{O}(X)$, then $x \wedge l_{uv}(y) = l_{uv}(x|_v \wedge y)$ holds for any $x \in F(u), y \in F(v)$ where $l_{uv} : F(v) \rightarrow F(u)$ is the left adjoint of the restriction map $F(u) \rightarrow F(v)$.

Corollary 2.3. *Let F be a complete posheaf on a locale X . The following conditions are equivalent:*

- (1) F is a complete Heyting sheaf.
- (2) F is a meet continuous distributive posheaf.

3 Continuous Posheaves

For a posheaf F on a locale X , we can construct an ideal-powersheaf $\mathcal{ID}F$ of F defined by $\mathcal{ID}F(u) = ID(F^u)$ where $ID(F^u)$ is the set of all ideals of F^u , and each restriction map $\mathcal{ID}F(u) \rightarrow \mathcal{ID}F(v)$ is same as the restriction map for $\mathbb{P}F$. So $\mathcal{ID}F$ is a subsheaf of $\mathbb{P}F$.

If p is a point of a posheaf F , the principle ideal $\downarrow p$ is clearly an ideal of F . Hence the principle ideal embedding morphism $\downarrow : F \rightarrow \mathbb{P}F$ can be factored through $\mathcal{ID}F$,

$$\downarrow : F \rightarrow \mathcal{ID}F$$

By Proposition 2.1, we have the following result.

Lemma 3.1. *For any posheaf F on a locale X , F is directed complete if and only if the principle ideal embedding $\downarrow : F \rightarrow \mathcal{ID}(F)$ has a left adjoint.*

Let F be a directed complete posheaf on X and $\text{sup}_F : \mathcal{ID}(F) \rightarrow F$ the left adjoint of the principle ideal embedding $\downarrow : F \rightarrow \mathcal{ID}(F)$. F is said to be a continuous posheaf if $\text{sup}_F : \mathcal{ID}(F) \rightarrow F$ has a left adjoint.

Proposition 3.1. *Let F be a directed complete posheaf on X . The follows are equivalent:*

- (1) F is a continuous posheaf;
- (2) for each global point $t \in F(1_X)$, there exists a smallest ideal I_t of F such that $\bigvee I_t = t$ and for each $u \in \mathcal{O}(X)$, the restriction I_t^u of I_t to F^u be the smallest ideal such that $\bigvee I_t^u = t|_u$;
- (3) $\sup_F : \mathcal{ID}(F) \rightarrow F$ is an inf-preserving morphism.

Corollary 3.1. *Let F be a continuous posheaf on X . Then for each $y \in F(u)$, $u \in \mathcal{O}(X)$, there exists $\min\{v \leq u \mid l_{uv}(y|_v) = y\}$ which we denoted by $u(y)$ with the property that for each $v \leq u$, $v(y|_v) = v \wedge u(y)$ where $l_{uv} : F(v) \rightarrow F(u)$ be the left adjoint of the restriction map $F(u) \rightarrow F(v)$.*

Proposition 3.2. *Let F be a continuous posheaf and G a directed complete posheaf on X . If $\alpha : F \rightarrow G$ is a surjective morphism which preserves directed sups and has a left adjoint then G is continuous.*

Proof Write $\beta : G \rightarrow F$ to be the left adjoint of α and $\gamma : F \rightarrow \mathcal{ID}(F)$ to be the left adjoint of the morphism $\sup_F : \mathcal{ID}(F) \rightarrow F$. We now show that the composite $\alpha_*\gamma\beta : G \rightarrow F \rightarrow \mathcal{ID}(F) \rightarrow \mathcal{ID}(G)$ is the left adjoint of the morphism $\sup_G : \mathcal{ID}(G) \rightarrow G$ where $\alpha_* : \mathcal{ID}(F) \rightarrow \mathcal{ID}(G)$ be the image morphism determined by α .

Suppose $u \in \mathcal{O}(X)$, and $y \in G(u)$. If $\alpha_{*u}(\gamma_u(\beta_u(y))) \subseteq J$ for some $J \in \mathcal{ID}(G)^u$, then $\sup_{Gu}(\alpha_{*u}(\gamma_u(\beta_u(y)))) \leq \sup_{Gu}(J)$. But $\sup_{Gu}(\alpha_{*u}(\gamma_u(\beta_u(y)))) = \alpha_u(\sup_{Fu}(\gamma_u(\beta_u(y)))) = \alpha_u(\beta\beta_u(y)) = y$ since α preserves directed sups and is a surjective morphism.

Conversely, if $y \leq \sup_{Gu}(J)$ for some $J \in \mathcal{ID}(G)^u$, then $\beta_u(y) \leq \beta_u(\sup_{Gu}(J)) = \sup_{Gu}(\beta_{*u}(J))$ since β preserves sups. Hence $\gamma_u(\beta_u(y)) \subseteq \beta_{*u}(J)$ by the adjointness. It implies that $\alpha_{*u}(\gamma_u(\beta_u(y))) \subseteq \alpha_{*u}(\beta_{*u}(J)) = J$. \square

Definition 3.1. Let F be a complete posheaf. If $\sup_F : \mathbb{D}(F) \rightarrow F$ has a left adjoint then we call F a completely distributive posheaf, where $\sup_F : \mathbb{D}(F) \rightarrow F$ is the left adjoint of the principle ideal embedding $F \rightarrow \mathbb{D}F$.

Proposition 3.3. *Let F be a complete posheaf on X . The follows are equivalent:*

- (1) F is a completely distributive posheaf;
- (2) For each global point $t \in F(1_X)$, there exists a lowersheaf G_t of F such that $\bigvee G_t = t$ and for each $u \in \mathcal{O}(X)$, the restriction G_t^u of G_t to F^u be the smallest lowersheaf such that $\bigvee G_t^u = t|_u$;
- (3) $\sup_F : \mathbb{D}(F) \rightarrow F$ is an inf-preserving morphism.

Lemma 3.2. *Every completely distributive posheaf is distributive.*

Proof Let F be a completely distributive posheaf on X . We only need to show that the global sections set $F(1_{\mathcal{O}(X)})$ is distributive.

Suppose $a, b, c \in F(1_{\mathcal{O}(X)})$, write $t = a \wedge (b \vee c)$. Denote G_t the smallest lowersheaf of F such that $\bigvee G_t = t$ and \widetilde{bc} be the lowersheaf of F generated by $\{b, c\}$. Then we have $G_t \subseteq \widetilde{bc}$ by the adjointness and the fact that $t \leq b \vee c = \bigvee \widetilde{bc}$. Hence $G_t \subseteq (a \wedge b)(a \wedge c)$ since $G_t \subseteq \downarrow a$, where $(a \wedge b)(a \wedge c)$ be the lowersheaf of F generated by $\{a \wedge b, a \wedge c\}$. This shows that $t = \bigvee G_t \leq \bigvee (a \wedge b)(a \wedge c) = (a \wedge b) \vee (a \wedge c)$. Thus $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. \square

Now we construct a left adjoint for the inclusion $\mathcal{IDF} \rightarrow \mathbb{D}F$.

For each $u \in \mathcal{O}(X)$ and a subsheaf G of F^u , consider the sub-presheaf \bar{G} of F^u defined by $\bar{G}(v) = \{l_{w_1 v}(x_1) \wedge \cdots \wedge l_{w_n v}(x_n) \mid x_i \in G(w_i), w_i \leq v, i = 1, \dots, n\}$ where $l_{w_i v} : G(w_i) \rightarrow G(v)$ be the left adjoint of the restriction map $G(v) \rightarrow G(w_i)$. Let \hat{G} be the ideal generated by \bar{G} . By Lemma 3.1 in [1], It is straightforward to show that the correspondence $\mathcal{DF}(u) \rightarrow \mathcal{IDF}, G \mapsto \hat{G}$ determines a functor which is left adjoint to the inclusion $\mathcal{IDF} \rightarrow \mathcal{DF}$.

We have the following diagram commutes

$$\begin{array}{ccc} \mathcal{IDF} & \xrightarrow{\quad} & \mathcal{DF} \\ & \searrow \text{sup} & \swarrow \text{sup} \\ & F & \end{array}$$

By the above arguments we have the following result.

Lemma 3.3. *Every completely distributive posheaf is a continuous posheaf.*

4 Continuous Frame Sheaves

Proposition 4.1. *Let X be a locale and let F be a complete posheaf on X . If F is continuous and distributive and for $v \leq u$ in $\mathcal{O}(X)$, we have $x \wedge l_{uv}(y) = l_{uv}(x|_v \wedge y)$ holds for any $x \in F(u), y \in F(v)$ where $l_{uv} : F(v) \rightarrow F(u)$ is the left adjoint of the restriction map $F(u) \rightarrow F(v)$. Then F is a frame sheaf.*

Proof We only need to show that each $F(u)$ is a frame for $u \in \mathcal{O}(X)$. Let $x \in F(u)$ and $D \subset F(u)$ be a directed set. Write $t = x \wedge \bigvee D$, denote I_t the smallest ideal on F^u such that $\bigvee I_t = t$ and \bar{D} the ideal on F^u generated by D . Then we have $I_t \subset \bar{D}$ by the adjointness. Hence for each $v \leq u$ and $y \in I_t(v)$ there exists a cover $v = \bigvee v_j$ such that $y|_{v_j} \leq d_j|_{v_j}$ for some $d_j \in D$. This implies that $y \leq x \wedge \bar{d}$ for some $\bar{d} \in \bar{D}$. Thus $t \leq \bigvee xD$. Note that the distributivity of F implies that each $F(u)$ is a distributive lattice, so $F(u)$ is a frame. \square

Let $f : Y \rightarrow X$ be a locale morphism. By the equivalence of the category of frame sheaves on X and the category of frames under $\mathcal{O}(X)$, we can regard $f : Y \rightarrow X$ as a

frame sheaf. Now it is natural to ask under what conditions $f : Y \rightarrow X$ corresponds to continuous frame sheaf.

Definition 4.1. Let $f : Y \rightarrow X$ be a locale morphism such that the corresponding frame homomorphism $f^* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ has a left adjoint $f^\sharp : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. $y, z \in \mathcal{O}(Y)$. We call that z is f -waybellow y , written as $z \ll_f y$, if and only if $z \leq f^* f^\sharp(y)$, and for all $u \in \mathcal{O}(X)$, any directed subset $D \subset \mathcal{O}(Y)$ with $y \wedge f^*(u) \leq \bigvee D$, there exists a cover $u \wedge f^\sharp(z) = \bigvee \{u_j \in \mathcal{O}(X) \mid j \in J\}$ such that for $\forall j \in J$, $z \wedge f^*(u_j) \leq d_j \wedge f^*(u_j)$ for some $d_j \in D$.

For a given locale $f : Y \rightarrow X$ over X , the above binary relation \ll_f is a generalization of the way-below relation \ll on $\mathcal{O}(Y)$. Indeed, If we take X to be the terminal locale, i.e. $\mathcal{O}(X) = 2$ be the initial frame, then $z \ll_f y$ if and only if $z \ll y$. Similar to the way-below relation, \ll_f relation has following properties.

Proposition 4.2. Let $f : Y \rightarrow X$ be a locale morphism such that the corresponding frame homomorphism $f^* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ has a left adjoint $f^\sharp : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. The following statement hold for all $x, y, z \in \mathcal{O}(Y)$:

- (1) $z \ll_f y$ implies $z \leq y$;
- (2) $x \leq z \ll_f y \leq t$ implies $x \ll_f t$;
- (3) $z \ll_f y$ and $x \ll_f y$ implies $x \vee z \ll_f y$;
- (4) $0 \ll_f x$;
- (5) If $f : Y \rightarrow X$ is an open morphism then $z \ll_f y$ implies $z \wedge f^*(u) \ll_f y \wedge f^*(u)$ for all $u \in \mathcal{O}(X)$.
- (6) Suppose $z \leq f^*(u)$ and $u = \bigvee_{i \in I} u_i$. If $z \wedge f^*(u_i) \ll_f y_i$ for each $i \in I$ then $z \ll_f \bigvee_{i \in I} y_i$.

Proof (1) Take $u = f^\sharp(y)$ then $y \wedge f^*(u) = y$. We have $f^\sharp(z) \leq u$ since $z \leq f^* f^\sharp(y) = f^*(u)$. Let $D = \{y\}$ then there exists a cover $f^\sharp(z) = \bigvee \{u_j \in \mathcal{O}(X) \mid j \in J\}$ such that for $\forall j \in J$, $z \wedge f^*(u_j) \leq y \wedge f^*(u_j)$. Hence $z \wedge \bigvee_{j \in J} f^*(u_j) \leq y \wedge \bigvee_{j \in J} f^*(u_j)$. This implies that $z = z \wedge f^* f^\sharp(z) = z \wedge f^*(\bigvee_{j \in J} u_j) \leq y \wedge f^*(\bigvee_{j \in J} u_j) \leq y$.

(2) Clear.

(3) Let $u \in \mathcal{O}(X)$ and $D \subset \mathcal{O}(Y)$ be a directed subset such that $y \wedge f^*(u) \leq \bigvee D$. There exists a cover $u \wedge f^\sharp(z) = \bigvee \{u_j \in \mathcal{O}(X) \mid j \in J\}$ such that for $\forall j \in J$, $z \wedge f^*(u_j) \leq d_j \wedge f^*(u_j)$ for some $d_j \in D$ and another cover $u \wedge f^\sharp(x) = \bigvee \{v_i \in \mathcal{O}(X) \mid i \in I\}$ such that for $\forall i \in I$, $x \wedge f^*(v_i) \leq d_i \wedge f^*(v_i)$ for some $d_i \in D$. Then $u \wedge f^\sharp(x \vee z) = \bigvee \{v_i \vee u_j \mid i \in I, j \in J\}$, and $(x \vee z) \wedge f^*(v_i \vee u_j) \leq ((z \vee d_i) \wedge f^*(v_i)) \vee ((x \vee d_j) \wedge f^*(u_j)) \leq (z \vee d_i \vee x \vee d_j) \wedge f^*(u_i \vee v_j) \leq (x \vee z \vee d_k) \wedge f^*(u_i \vee v_j)$ for some $d_k \in D$ since D is directed.

(4) Clear.

(5) Let $u \in \mathcal{O}(X)$. First $z \leq f^* f^\sharp(y)$ implies that $z \wedge f^*(u) \leq f^* f^\sharp(y) \wedge f^*(u) = f^* f^\sharp(y \wedge f^*(u))$. Suppose $y \wedge f^*(u) \wedge f^*(v) \leq \bigvee D$ for a directed subset $D \subset \mathcal{O}(Y)$ and

$v \in \mathcal{O}(X)$. There exists a cover $u \wedge v \wedge f^\sharp(z) = \bigvee \{v_j \in \mathcal{O}(X) \mid j \in J\}$ such that for $\forall j \in J$, $z \wedge f^*(v_j) \leq d_j \wedge f^*(v_j)$ for some $d_j \in D$. Thus $z \wedge f^*(u) \wedge f^*(v_j) \leq d_j \wedge f^*(v_j)$ for all $j \in J$.

(6) Write $y = \bigvee y_i$ and $z_i = z \wedge f^*(u_i)$ for $i \in I$. Suppose $y \wedge f^*(v) \leq \bigvee D$ for some $v \in \mathcal{O}(X)$ and a directed subset $D \subset \mathcal{O}(Y)$. Then for each $i \in I$, there exists a cover $f^\sharp(z_i) \wedge v = \bigvee_{j \in J_i} v_j^i$, such that for each $j \in J_i$, $z_i \wedge f^*(v_j^i) \leq d_j \wedge f^*(v_j^i)$ for some $d_j \in D$. It is clear that the cover $f^\sharp(z) \wedge v = \bigvee_{i \in I} \bigvee_{j \in J_i} v_j^i$ satisfies the condition. \square

Write $\Downarrow_f y = \{z \in \mathcal{O}(Y) \mid z \ll_f y\}$. By Proposition 3.2 (6), we have the following result.

Corollary 4.1. *Let $f : Y \rightarrow X$ be a locale morphism. The assignment $u \mapsto \Downarrow_f f^*(u)$, $u \in \mathcal{O}(X)$ defines a subsheaf of the frame sheaf $f : Y \rightarrow X$.*

Theorem 4.1. *Let $f : Y \rightarrow X$ be a locale morphism. The corresponding frame sheaf of f is a continuous frame sheaf if and only if $f : Y \rightarrow X$ is an open morphism and $y = \bigvee \{x \in \mathcal{O}(Y) \mid x \ll_f y\}$ for all $y \in \mathcal{O}(Y)$.*

Proof Suppose f corresponds to a continuous frame sheaf F . For each $y \in \mathcal{O}(Y)$, by corollary 3.1, there exists $\min\{v \in \mathcal{O}(X) \mid y \leq f^*(v)\}$, hence $f^* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ has a left adjoint f^\sharp . Also $f^\sharp(y \wedge f^*(u)) = f^\sharp(y) \wedge u$ for all $y \in \mathcal{O}(Y)$ and $u \in \mathcal{O}(X)$ by corollary 3.1.

For each $y \in \mathcal{O}(Y)$, there exists a smallest ideal $I(y)$ of F such that $\bigvee I(y) = y$ and for each $u \in \mathcal{O}(X)$, the restriction $I(y)^u$ of $I(y)$ to F^u be the smallest ideal such that $\bigvee I(y)^u = y \wedge f^*(u)$. Now we show that $z \ll_f y$ for each $z \in I(y)_p$. Suppose that $z \in I(y)_p$, it is clear that $z \leq f^* f^\sharp(y)$. Let $u \in \mathcal{O}(X)$ and $y \wedge f^*(u) \leq \bigvee D$ for a directed subset $D \subset \mathcal{O}(Y)$. Write \bar{D} for the ideal generated by D . Then we have $y \wedge f^*(u) \leq \bigvee \bar{D}$. Hence $I(y)^u \subset \bar{D}$. It implies that $z \wedge f^*(u) \in \bar{D}$. Thus there exists a cover $u \wedge f^\sharp(z) = \bigvee \{u_j \in \mathcal{O}(X) \mid j \in J\}$ such that for $\forall j \in J$, $z \wedge f^*(u_j) \leq d_j \wedge f^*(u_j)$ for some $d_j \in D$.

Conversely, suppose $f : Y \rightarrow X$ is an open morphism and $y = \bigvee \{x \in \mathcal{O}(Y) \mid x \ll_f y\}$ for all $y \in \mathcal{O}(Y)$. For $y \in \mathcal{O}(Y)$, by proposition 4.2, the set $\{x \in \mathcal{O}(Y) \mid x \ll_f y\}$ is directed. Write $I(y)$ for the ideal generated by the directed set $\{x \in \mathcal{O}(Y) \mid x \ll_f y\}$. Then it is not difficult to show that $I(y)$ is the smallest ideal of $f : Y \rightarrow X$ such that $\bigvee I(y) = y$ and for each $u \in \mathcal{O}(X)$, the restriction $I(y)^u$ of $I(y)$ be the smallest ideal such that $\bigvee I(y)^u = y \wedge f^*(u)$. \square

Now we consider the conditions that makes a frame sheaf to be a completely distributive frame sheaf.

Let $f : Y \rightarrow X$ be a locale morphism such that the corresponding frame homomorphism $f^* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ has a left adjoint $f^\sharp : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. $y, z \in \mathcal{O}(Y)$. We introduce a stronger binary relation on $\mathcal{O}(Y)$ as following.

We write $z \triangleleft_f y$, if and only if $z \leq f^* f^\sharp(y)$, and for all $u \in \mathcal{O}(X)$, any subset $B \subset \mathcal{O}(Y)$ with $y \wedge f^*(u) \leq \bigvee B$, there exists a cover $u \wedge f^\sharp(z) = \bigvee \{u_j \in \mathcal{O}(X) \mid j \in J\}$ such that for $\forall j \in J$, $z \wedge f^*(u_j) \leq b_j \wedge f^*(u_j)$ for some $b_j \in B$. Similar to the proof of proposition 4.2, we have the following result.

Lemma 4.1. *Let $f : Y \rightarrow X$ be a locale morphism such that the corresponding frame homomorphism $f^* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ has a left adjoint $f^\sharp : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. Then the following statements hold for all $x, y, z \in \mathcal{O}(Y)$:*

- (1) $z \triangleleft_f y$ implies $z \ll_f y$;
- (2) $x \leq z \triangleleft_f y \leq t$ implies $x \triangleleft_f t$;
- (3) $0 \triangleleft_f x$;
- (4) If $f : Y \rightarrow X$ is an open morphism then $z \triangleleft_f y$ implies $z \wedge f^*(u) \triangleleft_f y \wedge f^*(u)$ for all $u \in \mathcal{O}(X)$.
- (5) Suppose $z \leq f^*(u)$ and $u = \bigvee_{i \in I} u_i$. If $z \wedge f^*(u_i) \triangleleft_f y_i$ for each $i \in I$ then $z \triangleleft_f \bigvee_{i \in I} y_i$.

Similar to the proof of theorem 4.1, the following result is clear.

Theorem 4.2. *Let $f : Y \rightarrow X$ be a locale morphism. The corresponding frame sheaf of f is a completely distributive posheaf if and only if $f : Y \rightarrow X$ is an open morphism and $y = \bigvee \{x \in \mathcal{O}(Y) \mid x \triangleleft_f y\}$ for all $y \in \mathcal{O}(Y)$.*

Lemma 4.2. *If $f : Y \rightarrow X$ is an open inclusion then $y \triangleleft_f y$ for all $y \in \mathcal{O}(Y)$.*

Proof Suppose $f : Y \rightarrow X$ be an open inclusion then $f^* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ is surjective. Hence for each $u \in \mathcal{O}(X)$ and any subset $B \subset \mathcal{O}(Y)$ with $y \wedge f^*(u) \leq \bigvee B$, we have $u \wedge f^\sharp(y) = f^\sharp(y \wedge f^*(u)) \leq \bigvee f^\sharp(B)$. This cover satisfying $y \wedge f^* f^\sharp(b) = y \wedge b \leq b$ for all $b \in B$. \square

Proposition 4.3. *If $f : Y \rightarrow X$ is a local homeomorphism then it corresponds to a completely distributive posheaf, hence a continuous posheaf.*

Proof We first note that for each open inclusion $i_x : \downarrow x \rightarrow Y$, $z \triangleleft_{f i_x} y$ implies that $z \triangleleft_f y$ for all $z, y \leq x$ since $(f i_x)^\sharp(y) = f^\sharp(y)$ and $y \wedge f^*(u) = y \wedge (f i_x)^*(u)$ for all $u \in \mathcal{O}(X)$.

Suppose $\bigvee \{y_i \mid i \in I\} = 1_{\mathcal{O}(Y)}$ be a cover of Y such that each composite $f i_{y_j} : \downarrow y_j \rightarrow Y \rightarrow X$ is an open inclusion. For each $y \in \mathcal{O}(Y)$, by lemma 4.2, we have $y \wedge y_j \triangleleft_f y \wedge y_j$ for all $j \in I$. Hence $y = \bigvee \{x \in \mathcal{O}(Y) \mid x \triangleleft_f y\}$. \square

In general, a continuous posheaf is not a completely distributive posheaf. Indeed if we consider the case when X be the terminal locale then a continuous posheaf is just a continuous lattice and a completely distributive posheaf is just a completely distributive lattice.

Definition 4.2. Let $f : Y \rightarrow X$ be a locale morphism. $y \in \mathcal{O}(Y)$ is said to be a f -coprime element if $y \wedge f^*(u) \leq (x \vee z) \wedge f^*(u)$ implies that $y \wedge f^*(u) \leq x \wedge f^*(u)$ or $y \wedge f^*(u) \leq z \wedge f^*(u)$ for any $x, z \in \mathcal{O}(Y)$ and $u \in \mathcal{O}(X)$. Write CP_f for the set of all f -coprime elements of $\mathcal{O}(Y)$.

If X being the terminal locale, then an f -coprime element of $\mathcal{O}(Y)$ is just a coprime element in $\mathcal{O}(Y)$. If $\mathcal{O}(Y)$ is the powerset lattice of a set then every singleton set is an f -coprime element. If $\mathcal{O}(Y) = [0, 1]$ be the unit interval then every element of $\mathcal{O}(Y)$ is an f -coprime element.

Proposition 4.4. *Let $f : Y \rightarrow X$ be a locale morphism such that the corresponding frame sheaf of f is a continuous posheaf. If $\mathcal{O}(Y)$ has enough f -coprime elements, i.e. $x = \bigvee \{y \leq x \mid y \in CP_f\}$ for every $x \in \mathcal{O}(Y)$ then the corresponding frame sheaf of f is a completely distributive posheaf.*

Proof We only need to show that $z \in CP_f$ and $z \ll_f x$ implies that $z \triangleleft_f x$. It is clear by the definition. \square

Recall that an internal frame F in a localic topos $Sh(X)$ is called spatial if F is isomorphic to a subframe of a power object. Classically we know every continuous frame, moreover every completely distributive lattice is spatial under the assumption of AC. But we don't know whether it is still true in a localic topos. Under some additional conditions, we can show a partial result.

Let $f : Y \rightarrow X$ be a locale morphism such that the corresponding frame sheaf of f is a completely distributive posheaf. We call $f : Y \rightarrow X$ an algebraic completely distributive posheaf if $y = \bigvee \{x \in \mathcal{O}(Y) \mid x \triangleleft_f x \leq y\}$ for all $y \in \mathcal{O}(Y)$.

Proposition 4.5. *If $f : Y \rightarrow X$ is an algebraic completely distributive posheaf then $f : Y \rightarrow X$ is spatial.*

Proof Write $\Downarrow_f y = \bigvee \{x \in \mathcal{O}(Y) \mid x \triangleleft_f x \leq y\}$ for $y \in \mathcal{O}(Y)$. By Lemma 3.1, $G(u) = \Downarrow_f f^*(u)$ defines a subsheaf of the frame sheaf $f : Y \rightarrow X$. We now show that the frame sheaf $f : Y \rightarrow X$ can be embed as a subsheaf into the powersheaf $\mathbb{P}G$.

Indeed, consider the morphism $\gamma : F \rightarrow \mathbb{P}G$ such that for each $u \in \mathcal{O}(X)$ and $y \leq f^*(u)$,

$$\gamma_u(y)(v) = \begin{cases} \Downarrow_f (y \wedge f^*(v)), & v \leq f^\#(y) \\ \emptyset, & v \not\leq f^\#(y) \end{cases}$$

Then it is straightforward to show that γ is a frame monomorphism.

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